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OPTIMAL CAR OWNERSHIP UNDER CONDITIONS OF ROAD CAPACITY CONSTRAINTS

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Abstract

This paper presents a bi-level optimization model of car ownership problem. An optimal car ownership is determined under conditions of road capacity constraints. The lower-level problem is a combined trip distribution/assignment (CDA) model, while the upper-level problem is to maximize the sum of the number of cars by traffic zone. A sensitivity analysis based algorithm is developed and illustrated with a numerical example.

INTRODUCTION

Car ownership growth has been the dominant structural feature of the development of the transport sector for the past three decades. Household car ownership and utilization patterns play an important role in traffic management and transport planning. A car availability model exists as part of the Hong Kong transport model which takes as input income distribution, public transport accessibility, residential parking supply, and car ownership and usage costs. The discrepancy between estimated car ownership and actual car ownership is partly a result of not considering the effects of constraints of the road capacities. Therefore, the response of car owners and/or road users to these constraints should be investigated. There is a need to review the current approach for prediction of the car ownership in the light of both supply and demand sides.

The traditional methods for estimating car ownership are mainly concerned from the view of the users' desire. Car ownership is usually modelled as a function of household characteristics, socio-economic variables and/or public transport services (Ben-Akiva *et al.*, 1981; Jansson, 1989; Pendyala *et al.*, 1995). The joint models of car ownership and use (Train, 1986; Train and Lohrer, 1982), and simultaneous equations systems of car ownership, mechanized trip generation and modal split (Kitamura, 1987, 1988, 1989) are developed and applied to simulate future car ownership and mode use. However, the effect of road traffic condition (e.g. the constraint of road capacity) is not to be considered for the forecast of car ownership. This would lead to the problem of traffic congestion as the network constraints would affect the car usage and in turn the desire of car ownership.

In this paper, an optimal car ownership under conditions of road capacity constraints is estimated by a bi-level optimization model (Suh *et al.* 1990; Yang and Lam 1996). In other words, the reserve capacity for accommodation of the additional number of cars in the traffic zones can be determined by the model. The model is formulated as a bi-level programming problem including car ownership estimation, trip distribution and traffic assignment. The lower-level problem is an combined equilibrium trip distribution/assignment (CDA) model (Evans, 1976; Lam and Huang, 1992), while the upper-level problem is to maximize the sum of the number of cars by traffic zone.

The CDA model is adopted to incorporate a destination attractiveness measure reflecting the activity opportunities available there, and determine the destination and route choices of travellers simultaneously for any given number of trips originating from each origin by solving an equivalent convex programming problem. In the proposed model, the number of trips generating from the origin zone is expressed as a function of the number of cars owned by residents in that zone. The optimal car ownership is estimated by considering the route and destination choice behaviour of travellers and satisfying road capacity constraints. A heuristic sensitivity analysis based algorithm is described in the paper. A numerical example is presented to illustrate the bi-level model and the sensitivity analysis method.

NOTATION

Sets

A:	the set of links in the road network
W:	the set of Origin-Destination (O-D) pairs
R:	the set of paths in the road network
R_w:	the set of paths between O-D pair $w \in W$

I: the set of origin nodes (trip-producing zones)
J: the set of destination nodes (trip-attracting zones)

Vectors/ Matrices

v: a vector of link flows (lower-level decision variables)
f: a vector of path flows
c(**v**): a vector of link travel time functions
$\mathbf{c}^{+}(\mathbf{f})$: a vector of path travel time functions
T: a vector of O-D demand by car
O:. a vector of trip productions
D: a vector of trip attractions
u: a vector of the number of cars (upper-level decision variables)
Λ: O-D/path incidence matrix where entries δ_{wr} are 1 if path $r \in R_w$, and 0 otherwise
Δ: link/path incidence matrix where entries δ_{ar} are 1 if path $r \in R$, and 0 otherwise

Constants

S_a: capacity (pcu/hr) of link $a \in A$
C_a: free-flow travel time (hrs) of link $a \in A$
r_i: trip generation rate of each car (car trips/hr/pcu) in node i
α: a dispersion parameter for a gravity-type trip distribution model
$\bar{u}_i$: lower limit of the number of cars in node i
$\hat{u}_i$: upper limit of the number of cars in node i

Variables

v_a: flow (pcu/hr) on link $a \in A$
f_r: flow (pcu/hr) on path $r \in R$
$c_a(v_a)$: travel time (hrs) on link $a \in A$
t_{ij}: travel demand between O-D pair (i,j)
O_i: trip production by car (car trips/hr) at origin node i
D_j: trip attraction by car (car trips/hr) at destination node j
u_i: the number of cars in node i

MODEL FORMULATION

The problem to maximize the sum of car ownership by traffic zone under the condition of road capacity constraints, can be formulated as the following bi-level optimization model:

$$\underset{\mathbf{u}}{Maximize} \sum_i u_i \tag{1}$$

subject to

$$v_a(\mathbf{u}) \le S_a, \quad a \in A \tag{2}$$

$$\sum_{i \in I} O_i(u_i) = \sum_{j \in J} D_j \tag{3}$$

$$\bar{u}_i \le u_i \le \hat{u}_i, \quad i \in I \tag{4}$$

where the equilibrium flow $v_a(\mathbf{u})$, $a \in A$ is obtained by solving the following network equilibrium trip distribution and assignment (CDA) problem:

$$\underset{\mathbf{v}}{Minimize} \sum_a \int_0^{v_a} c_a(x)dx + \frac{1}{\alpha}\sum_i \sum_j t_{ij}(ln\, t_{ij} - 1) \tag{5}$$

subject to

$$\sum_{r \in R_w} f_r = t_{ij}, i \in I, j \in J \tag{6}$$

$$\sum_{j \in J} t_{ij} = O_i(u_i), i \in I \tag{7}$$

$$\sum_{i \in I} t_{ij} = D_j, j \in J \tag{8}$$

$$v_a = \sum_{r \in R} f_r \delta_{ar}, a \in A \tag{9}$$

$$f_r \geq 0, r \in R \tag{10}$$

$$t_{ij} \geq 0, i \in I, j \in J \tag{11}$$

The maximum number of cars owned by residents in each zone is referred as the zonal car ownership growth potential. i.e. the upper bound of zonal car ownership can be accommodated subject to the traffic flow without exceeding the capacity of any link.

The lower-level problem (5)-(11) is a standard combined trip distribution and assignment problem that can be solved by a convex-combination method for given car ownership **u**. The traffic flow v_a so obtained represents the equilibrium flow on link $a \in A$ when the number of cars in the origin zone i is u_i. These results are then fed into the upper-level problem (1)-(4) to solve for the maximum total car ownership that constrained by the link capacities and met the total travel demands in the road network. In general, the number of cars in each zone should be bounded by the lower and upper limits. The lower bound for car ownership is set to retain certain mobility in that zone. The upper bound for car ownership is due to the land use of that zone.

A new set of car ownership u_i will be obtained from solving the upper-level problem. This set of car ownership will then be applied to the constraint (7) of the lower-level problem to provide the new values of trip production, while the trip attraction is assumed to be constant. The function of trip productions is simply defined as the multiplication of trip generation rate per car and the number of cars in each zone; i.e. $O_i(u_i) = r_i u_i$. This process will be repeated until a desirable convergence is achieved.

SENSITIVITY ANALYSIS

Firstly, a general situation that perturbation parameters exist in the link travel time function, O-D demands, trip productions and trip attractions is considered. For convenience of exposition, the equations are used with vector and matrix notation. The perturbed problem can be written as

$$\underset{\mathbf{v}}{Minimize} \sum_a \int_0^{v_a} c_a(x,\varepsilon)dx + \frac{1}{\alpha}\sum_i \sum_j t_{ij}(\varepsilon)(ln\, t_{ij}(\varepsilon) - 1) \tag{12}$$

subject to

$$\Lambda\mathbf{f} = \mathbf{T} \tag{13}$$

$$\Phi_i\mathbf{T} = \mathbf{O}(\varepsilon) \tag{14}$$

$$\Phi_j\mathbf{T} = \mathbf{D}(\varepsilon) \tag{15}$$

$$\mathbf{v} = \Delta\mathbf{f} \tag{16}$$

$$\mathbf{f} \geq 0 \tag{17}$$

$$t_{ij}(\varepsilon) \geq 0 \tag{18}$$

where ε is a vector of perturbation parameters, Φ_i and Φ_j can be defined as follows for a one-way road network, while m is the number of origin nodes and n is the number of destination nodes.

$$\Phi_i = \begin{bmatrix} \underbrace{1\cdots1}_{n} & 0 & \cdots & \cdots & \cdots & \cdots 0 \\ 0\cdots0 & \underbrace{1\cdots1}_{n} & 0 & \cdots & \cdots & \cdots 0 \\ 0 & \cdots & \cdots 0 & \underbrace{1\cdots1}_{n} & 0 \cdots & \cdots 0 \\ \vdots & \cdots & \cdots & \cdots 0 & \ddots & 0\cdots0 \\ 0 & \cdots & \cdots & \cdots & \cdots 0 & \underbrace{1\cdots1}_{n} \end{bmatrix}_{m\times mn}, \tag{19}$$

$$\Phi_j = \begin{bmatrix} \underbrace{10\cdots0}_{n} & \underbrace{10\cdots0}_{n} & \cdots\cdots & \underbrace{10\cdots0}_{n} \\ \underbrace{010\cdots0}_{n} & \underbrace{010\cdots0}_{n} & \cdots\cdots & \underbrace{010\cdots0}_{n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \underbrace{0\cdots010\cdots0}_{n} & \underbrace{0\cdots010\cdots0}_{n} & \cdots\cdots & \underbrace{0\cdots010\cdots0}_{n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \underbrace{0\cdots01}_{n} & \underbrace{0\cdots01}_{n} & \cdots\cdots & \underbrace{0\cdots01}_{n} \end{bmatrix}_{n\times mn}, \tag{20}$$

and

$$\mathbf{O} = \begin{bmatrix} O_1(\varepsilon) \\ \vdots \\ O_i(\varepsilon) \\ \vdots \\ O_m(\varepsilon) \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} D_1(\varepsilon) \\ \vdots \\ D_j(\varepsilon) \\ \vdots \\ D_n(\varepsilon) \end{bmatrix}. \tag{21}$$

It is assumed that $c_a(v_a,\varepsilon)$, $t_{ij}(\varepsilon)$, $O_i(\varepsilon)$ and $D_j(\varepsilon)$ are once continuously differentiable in ε. The solution of the perturbed problem for $\varepsilon^* = 0$ is assumed to be $\mathbf{v}^*(0)$ and $\mathbf{f}^*(0)$ and that $c_a(v_a,\varepsilon)$ is strongly monotone in v_a so that the solution is unique.

It is noted that the direct application of the standard sensitivity analysis (Fiacco, 1976) to the perturbed combined trip distribution and traffic assignment problem of eqns (12)-(18) is not feasible. It is because the path flow solution does not satisfy the uniqueness condition even if the link flow solution is unique. In order to overcome the non-uniqueness difficulty, the restricted network equilibrium approach proposed by Tobin and Friesz (1988) is adopted to derive the derivative expressions.

This approach is to select a nondegenerate extreme point in the feasible region of equilibrium path flows. An extreme point can be obtained easily if the convex combination method (Frank-Wolfe method) suggested by LeBlanc *et al.* (1975) is used to solve the equilibrium assignment problem. The Frank-Wolfe algorithm generates a unique set of minimum time paths by O-D pairs at each iteration. If the paths generated are saved from iteration to iteration, upon termination the Frank-Wolfe algorithm provides an equilibrium path flow pattern and a link/path incidence matrix for the paths used. An extreme point in the feasible region can then be identified from this set of equilibrium path flows (Yang and Yagar, 1994; Yang *et al.*, 1994).

Let $\mathbf{f}^* > 0$ be a nondegenerate extreme point in the region of equilibrium path flows. It is easily observed that the necessary conditions for the perturbed equilibrium assignment problem of eqns (12)-(18) at $\varepsilon = 0$ lead to a solution for the following system equations:

$$\mathbf{c}^+(\mathbf{f}^*,0) + \frac{1}{\alpha}\Lambda^{\mathrm{T}}\ln(\Lambda\mathbf{f}^*) - (\Phi_i\Lambda)^{\mathrm{T}}\lambda - (\Phi_j\Lambda)^{\mathrm{T}}\gamma - \pi = 0 \tag{22}$$

$$\pi^{\mathrm{T}}\mathbf{f}^* = 0 \tag{23}$$

$$\Phi_i\Lambda\mathbf{f}^* - \mathbf{O}(0) = 0 \tag{24}$$

$$\Phi_j\Lambda\mathbf{f}^* - \mathbf{D}(0) = 0 \tag{25}$$

$$\pi \geq 0,\ \mathbf{f}^* \geq 0 \tag{26}$$

where λ, γ and π are the Lagrange multipliers of the eqns (14), (15) and (17) respectively.

It can be shown that under the assumption of strictly positive link flows and strictly complementary slackness, eqns (23) and (26) can be eliminated without changing the solution near $\varepsilon = 0$. Therefore, only the nondegenerate extreme points of the positive path flow solutions are considered. The system of eqns (22)-(26) can then be reduced to

$$\mathbf{c}^+(\mathbf{f}^{0*},0) + \frac{1}{\alpha}\Lambda^{0\mathrm{T}}\ln(\Lambda^0\mathbf{f}^{0*}) - (\Phi_i\Lambda^0)^{\mathrm{T}}\lambda - (\Phi_j\Lambda^0)^{\mathrm{T}}\gamma = 0 \tag{27}$$

$$\Phi_i\Lambda^0\mathbf{f}^{0*} - \mathbf{O}(0) = 0 \tag{28}$$

$$\Phi_j\Lambda^0\mathbf{f}^{0*} - \mathbf{D}(0) = 0 \tag{29}$$

where ^{0} represents the corresponding reduced vectors and matrices.

Let $$\mathbf{c}^+(\mathbf{f}^{0*},0) + \frac{1}{\alpha}\Lambda^{0\mathrm{T}}\ln(\Lambda^0\mathbf{f}^{0*}) = \mathbf{c}'(\mathbf{f}^{0*},0)\,, \tag{30}$$

$$\mathbf{h} = \begin{bmatrix}\lambda\\ \gamma\end{bmatrix},\ \mathbf{s}(\varepsilon) = \begin{bmatrix}\mathbf{O}(\varepsilon)\\ \mathbf{D}(\varepsilon)\end{bmatrix},\ \mathbf{M} = \begin{bmatrix}\Phi_i\Lambda^0\\ \Phi_j\Lambda^0\end{bmatrix}. \tag{31}$$

Then the system of eqns (27)-(29) can be written as

$$\mathbf{c}'(\mathbf{f}^{0*},0) - \mathbf{M}^{\mathrm{T}}\mathbf{h} = 0 \tag{32}$$

$$\mathbf{M}\mathbf{f}^{0*} - \mathbf{s}(0) = 0 \tag{33}$$

The Jacobian matrix of the system of eqns (32) and (33) with respect to $(\mathrm{f}^0, \lambda, \gamma)$ and evaluated at $\varepsilon =$ 0 is

$$\mathbf{J}_{\mathbf{f}^0,\lambda,\gamma} = \begin{bmatrix}\nabla_{\mathbf{f}}\mathbf{c}'(\mathbf{f}^{0*},0) & -\mathbf{M}^{\mathrm{T}}\\ \mathbf{M} & 0\end{bmatrix}. \tag{34}$$

Suppose that

$$\left[\mathbf{J}_{\mathbf{f}^0,\lambda,\gamma}\right]^{-1} = \begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix} \tag{35}$$

The following can be obtained

$$\mathbf{B}_{11} = \nabla_f \mathbf{c}'(\mathbf{f}^{0*},0)^{-1}[\mathbf{E} - \mathbf{M}^T[\mathbf{M}\nabla_f \mathbf{c}'(\mathbf{f}^{0*},0)^{-1}\mathbf{M}^T]^{-1}\mathbf{M}\nabla_f \mathbf{c}'(\mathbf{f}^{0*},0)^{-1}] \quad (36)$$
$$\mathbf{B}_{12} = \nabla_f \mathbf{c}'(\mathbf{f}^{0*},0)^{-1}\mathbf{M}^T[\mathbf{M}\nabla_f \mathbf{c}'(\mathbf{f}^{0*},0)^{-1}\mathbf{M}^T]^{-1} \quad (37)$$
$$\mathbf{B}_{21} = -[\mathbf{M}\nabla_f \mathbf{c}'(\mathbf{f}^{0*},0)^{-1}\mathbf{M}^T]^{-1}\mathbf{M}\nabla_f \mathbf{c}'(\mathbf{f}^{0*},0)^{-1} \quad (38)$$
$$\mathbf{B}_{22} = [\mathbf{M}\nabla_f \mathbf{c}'(\mathbf{f}^{0*},0)^{-1}\mathbf{M}^T]^{-1} \quad (39)$$

where $\mathbf{E}$ is an identity matrix of appropriate dimension, and

$$\nabla_f \mathbf{c}'(\mathbf{f}^{0*},0) = \nabla_f \mathbf{c}^{+}(\mathbf{f}^{0*},0) + \frac{1}{\alpha}\Lambda^{0T}(\Lambda^0\mathbf{f}^{0*})^{-1}\Lambda^0$$
$$= \Delta^{0T}\nabla_v \mathbf{c}(\mathbf{v}^*,0)\,\Delta^0 + \frac{1}{\alpha}\Lambda^{0T}(\Lambda^0\mathbf{f}^{0*})^{-1}\Lambda^0 \quad (40)$$

The Jacobian matrix of the system of eqns (32) and (33) with respect to ε and evaluated at ε = 0 is

$$\mathbf{J}_\varepsilon = \begin{bmatrix} \nabla_\varepsilon \mathbf{c}'(\mathbf{f}^{0*},0) \\ -\nabla_\varepsilon \mathbf{s}(0) \end{bmatrix}. \quad (41)$$

It can be shown that the Jacobian matrix $\mathbf{J}_{\mathbf{f}^0,\lambda,\gamma}$ is non-singular and the partial derivatives of $[\mathbf{f}^{0*},\lambda^*,\gamma^*]$ with respect to ε are given by

$$\begin{bmatrix} \nabla_\varepsilon \mathbf{f}^0 \\ \nabla_\varepsilon \mathbf{h} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} -\nabla_\varepsilon \mathbf{c}'(\mathbf{f}^{0*},0) \\ \nabla_\varepsilon \mathbf{s}(0) \end{bmatrix} \quad (42)$$

Therefore, the derivatives of path flows and Lagrange multipliers with respect to ε at ε = 0 are

$$\nabla_\varepsilon \mathbf{f}^0 = -\mathbf{B}_{11}\nabla_\varepsilon \mathbf{c}'(\mathbf{f}^{0*},0) + \mathbf{B}_{12}\nabla_\varepsilon \mathbf{s}(0) \quad (43)$$

and

$$\nabla_\varepsilon \mathbf{h} = -\mathbf{B}_{21}\nabla_\varepsilon \mathbf{c}'(\mathbf{f}^{0*},0) + \mathbf{B}_{22}\nabla_\varepsilon \mathbf{s}(0) \quad (44)$$

Since

$$\nabla_\varepsilon \mathbf{v} = \Delta^0\nabla_\varepsilon \mathbf{f}^0 \quad (45)$$

the derivatives of link flows with respect to ε at ε = 0 are obtained as

$$\nabla_\varepsilon \mathbf{v} = -\Delta^0\mathbf{B}_{11}\nabla_\varepsilon \mathbf{c}'(\mathbf{f}^{0*},0) + \Delta^0\mathbf{B}_{12}\nabla_\varepsilon \mathbf{s}(0) \quad (46)$$

where $\mathbf{B}_{11}$, $\mathbf{B}_{12}$, $\mathbf{B}_{21}$, $\mathbf{B}_{22}$ are given in eqns (36)-(39).

Eqns (46) and (44) are the general expressions of the derivatives of the decision variables (link flows) and constraint multipliers with respect to a variety of perturbation parameters in the network equilibrium problem.

The explicit expressions for link flows with respect to the number of cars by zone can be derived in the following. Since the link travel time functions and entropy term are fixed in the problem,

$$\nabla_\varepsilon \mathbf{c}'(\mathbf{f}^{0*},0) = 0 \quad (47)$$

From eqn (47), eqn (46) can be simplified into

$$\nabla_\varepsilon \mathbf{v} = \Delta^0 \mathbf{B}_{12} \nabla_\varepsilon \mathbf{s}(0) \tag{48}$$

Let $\varepsilon = \delta\mathbf{u}$, which represents a small variation in the number of cars, then the derivatives of link flows with respect to the number of cars by zone can be obtained.

From eqn (48),

$$\begin{aligned}
\nabla_\mathbf{u}\mathbf{v} &= \nabla_\varepsilon \mathbf{v} = \Delta^0 \mathbf{B}_{12} \nabla_\varepsilon \mathbf{s}(0) \\
&= \Delta^0 \mathbf{B}_{12} \begin{bmatrix} \nabla_\varepsilon \mathbf{O}(0) \\ \nabla_\varepsilon \mathbf{D}(0) \end{bmatrix}, \text{ from eqn (31)} \\
&= \Delta^0 \mathbf{B}_{12} \begin{bmatrix} \nabla_\mathbf{u} \mathbf{O} \\ \nabla_\mathbf{u} \mathbf{D} \end{bmatrix} \\
&= \Delta^0 \nabla_\mathbf{f} \mathbf{c}'(\mathbf{f}^{0*},0)^{-1} \mathbf{M}^T \mathbf{B}_{22} \begin{bmatrix} \nabla_\mathbf{u} \mathbf{O} \\ \nabla_\mathbf{u} \mathbf{D} \end{bmatrix}, \text{ from eqns (37) and (39)} \\
&= \Delta^0 \left[\Delta^{0T} \nabla_\mathbf{v} \mathbf{c}(\mathbf{v}^*,0) \Delta^0 + \frac{1}{\alpha} \Lambda^{0T} (\Lambda^0 \mathbf{f}^{0*})^{-1} \Lambda^0 \right]^{-1} \mathbf{M}^T \mathbf{B}_{22} \begin{bmatrix} \nabla_\mathbf{u} \mathbf{O} \\ \nabla_\mathbf{u} \mathbf{D} \end{bmatrix}, \text{ from eqn (40)}
\end{aligned} \tag{49}$$

where $\mathbf{B}_{22}$ are given by eqn (39).

In our problem, $O_i(u_i) = r_i u_i$ and D_j is a constant for $i \in I, j \in J$. Therefore,

$$\nabla_\mathbf{u} \mathbf{O} = \begin{bmatrix} r_1 & 0 & \cdots & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & r_m \end{bmatrix}_{m \times m} \quad \text{and} \quad \nabla_\mathbf{u} \mathbf{D} = \mathbf{0}_{n \times m}. \tag{50}$$

SOLUTION ALGORITHM

The derivatives of link flows with respect to the number of cars by zone are obtained from the theory of sensitivity analysis for a given solution of the network equilibrium problem. This derivative information is essential in the development of the solution algorithm for the proposed bi-level car ownership problem. It is because the capacity constraints (2) in the upper-level problem involve the nonlinear and implicit functions of control variable $\mathbf{u}$. Therefore, local linear approximations of capacity constraints based on the derivatives of $\mathbf{v}(\mathbf{u})$ with respect to $\mathbf{u}$ is used to solve the bi-level problem. The resulting linear programming problem can then be solved using the well-known simplex method.

The linear approximation of capacity constraints (2) can be derived as follows:

$$v_a(\mathbf{u}) \approx v_a(\mathbf{u}^*) + \nabla_\mathbf{u} v_a(\mathbf{u}^*)(u_i - u_i^*), \ a \in A \tag{51}$$

where $\nabla_\mathbf{u} v_a$ is given by eqn (49) and is the derivative of link flow v_a, $a \in A$ with respect to the number of cars by zone u_i, $i \in I$. $\mathbf{u}^*$ is the initial solution and $\mathbf{v}_a(\mathbf{u}^*)$ is the corresponding equilibrium link flow pattern. From eqn (51), the capacity constraints (2) thus become

$$\nabla_{\mathbf{u}} v_a(\mathbf{u}^*) u_i \leq S_a - v_a(\mathbf{u}^*) + \nabla_{\mathbf{u}} v_a(\mathbf{u}^*) u_i^*, \; a \in A \tag{52}$$

which is a set of simple linear constraints.

The mechanism of the solution algorithm is an iterative process between the upper-level and lower-level problems. The proposed sensitivity analysis based (SAB) algorithm can be described as follows:

Step 0. Determine an initial car ownership values $\mathbf{u}^{(k)}$. Set $k = 0$.

Step 1. Solve the lower-level combined trip distribution and assignment problem (5)-(11) for given $\mathbf{u}^{(k)}$ and hence get $\mathbf{v}^{(k)}$.

Step 2. Calculate the derivatives $\nabla_{\mathbf{u}}\mathbf{v}^{(k)}$ using sensitivity analysis method.

Step 3. Formulate local linear approximations of the upper-level capacity constraint (2) using the derivative information, and solve the resulting linear programming problem to obtain an auxiliary solution $\mathbf{y}$.

Step 4. Compute a new number of cars $\mathbf{u}^{(k+1)}$ by the method of successive average (MSA).

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \frac{(\mathbf{y} - \mathbf{u}^{(k)})}{k+1} \tag{53}$$

Step 5. If $|u_i^{(k+1)} - u_i^{(k)}| \leq \omega$ for all $i \in I$ or k = MK then stop, where ω is a predetermined error tolerance and MK is the maximum number of iterations. Otherwise let $k := k+1$ and return to Step 1.

It should be noted that the number of cars resulted in the MSA may not be an integer number, but the passenger car units (pcu) are used in this paper for the unit of car ownership (including motorcycles, small vans and light goods vehicles).

NUMERICAL EXAMPLE

A numerical example is presented to illustrate how to use the proposed method to obtain the optimal number of cars under conditions of road capacity constraints. The example road network shown in Figure 1, consists of 7 links, 6 nodes and 2 O-D pairs (of which 1 and 2 are origin nodes and 5 and 6 are destination nodes). The BPR (Bureau of Public Roads, 1964) link travel time function is used with associated input data given in Table 1.

$$c_a(v_a) = C_a\{1.0 + 0.15(\frac{v_a}{S_a})^4\} \tag{54}$$

The trip generation rate in origin node 1 is assumed to be 2 car trips/hr/pcu and in origin node 2 is 3 car trips/hr/pcu. The trip attractions are given as $\mathbf{D}$ = [120, 90], which is the total number of parking spaces in each destination node. The lower and upper bounds of car ownership are assumed to be [10, 100] in origin node 1 and [10, 80] in origin node 2. The lower bound is set to retain at least 10 pcu mobility on private vehicles for travelling and goods transportation, rather than only depends on public transport. The upper bound is set for considering the amount of vacant land that is suitable and available for residential use. The value of dispersion parameter α is assumed to be 0.1 for the gravity-type trip distribution model in this example.

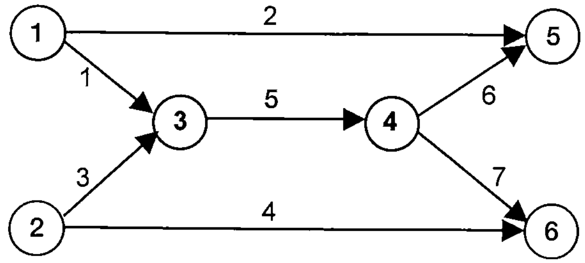

Figure 1 - Example Road Network

Table 1 - Link Travel Time Data for the Network

Link Number a	Free-flow Travel Time C_a (minutes)	Capacity S_a (pcu/hr)
1	4	60
2	10	80
3	4	70
4	10	80
5	5	110
6	4	70
7	4	60

The initial number of cars is set to be $\mathbf{u}^{(0)}$ = [30, 50] and the resultant link flows are $\mathbf{v}^{(0)}$ = [25.71, 34.29, 85.71, 64.29, 111.43, 85.71, 25.71]. It can be seen that the traffic flows on links 3, 5 and 6 are greater than their road capacities, while the remaining link flows are satisfied the road capacity constraints. Therefore, it is a potential to accommodate future growth on car ownership and in turn an increase in traffic flows. However, the number of cars at origin zone(s) should be suppressed to a certain level due to the violation of the road capacity constraints (Links 3, 5 and 6). Hence, the proposed model can be applied to obtain the balance on both sides.

The numerical results of the proposed model with different error tolerances are summarized in Table 2. The convergence is achieved in 230 iterations at the error tolerance $\omega = 1.0\times10^{-4}$. Figure 2 shows the changes of optimal car ownership versus the number of iterations. The optimal number of cars is found in each origin node. The maximum number of cars in the study area is 93.33 pcu which is greater than the initial figure 80 pcu. The number of cars owned by residents in zone 1 is increased from 30 pcu to 69.98 pcu, while that in zone 2 is reduced from 50 pcu to 23.35 pcu. Hence, the traffic congestion made by the initial case was due to the exceeding number of cars owned by residents in zone 2. The corresponding link flows and O-D demand matrix are given in Tables 3 and 4. As shown in Table 3, links 1, 2 and 7 are identified to become saturated when car ownership equals to the estimated maximum value in each of the two origin zones. Thus capacity improvement is required in these links if more cars are to be owned by residents in the origin traffic zones.

Table 2 - Numerical Results obtained by the SAB Algorithm

Error Tolerance ω	Number of Iterations	Final Solution u_1	Final Solution u_2
1.0×10^{-1}	8	69.40	23.73
1.0×10^{-2}	23	69.78	23.48
1.0×10^{-3}	72	69.93	23.38
1.0×10^{-4}	230	69.98	23.35

Figure 2 - Convergence of Optimal Car Ownership at the Error Tolerance ω = 1.0×10^{-4}

Table 3 - The Corresponding Equilibrium Link Flow Pattern

Link Number a	Equilibrium Link Flow v_a (pcu/hr)	Flow/Capacity Ratio v_a/S_a
1	59.98	1.00
2	79.97	1.00
3	40.03	0.57
4	30.02	0.38
5	100.01	0.91
6	40.03	0.57
7	59.98	1.00

Table 4 - Estimated Origin-Destination (O-D) Matrix

		Destination nodes 5	6	O_i
Origin Nodes	1	79.97	59.98	139.95
	2	40.03	30.02	70.05
	D_j	120.00	90.00	210.00

The predicted maximum number of cars indicates to what extent zonal car ownership growth could be accommodated or suppressed by the existing transportation facilities. When compared with the existing number of cars, the additional amount is referred as the reserve capacity for accommodation of the number of cars in a traffic zone. In the above example, positive car ownership growth potential of 69.98 pcu exists in zone 1; while negative potential of 23.35 pcu occurs in zone 2. The reserve capacity of car ownership at traffic zone 1 is 39.98 pcu, while 26.65 pcu should be reduced in traffic zone 2. In terms of percentage, 133.27% of existing cars (in pcu) can be accommodated by the reserve capacity in zone 1. However, 53.30% of the existing amount (in pcu) should be suppressed in zone 2. Totally 13.33 pcu can gain in the two origin zones. Figure 3 shows the existing number of cars and the maximum number of cars in the two origin zones estimated by the model. The predicted maximum number of cars by zones would provide important information for planning of future transport infrastructure development and policies.

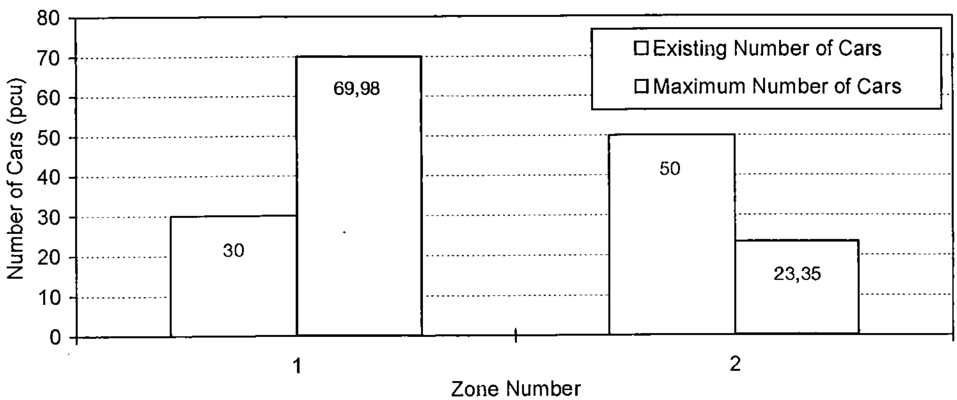

Figure 3 - Existing and Maximum Number of Cars in Origin Zones

CONCLUSIONS

This paper proposes a bi-level optimization model to maximize the number of cars in each of the traffic zones, subject to the network capacity constraints. It can examine whether existing transport network is capable of accommodating future car ownership growth and hence establishing efficient policies for controlling car ownership and improving road networks. The proposed model takes into account the route and destination choices of travellers, and the effects of car ownership by traffic zone on the trip generation. A sensitivity analysis based algorithm has also been developed for determining the maximum number of cars in the study area. The numerical example is presented to illustrate the application of the proposed model. Further work will continue to formulate the trip production and attraction as a function of car ownership by traffic zone and average journey time in the road network. Parking space and charges, and socio-economic factors that affect car ownership will also be incorporated in the model.

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