MINIMAL REVENUE NETWORK TOLLING: SYSTEM OPTIMISATION UNDER STOCHASTIC ASSIGNMENT

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Abstract

The classical road tolling problem is to toll network links such that under the principles of Wardropian User Equilibrium (UE) assignment, a System Optimising (SO) flow pattern is obtained. Such toll sets are however non-unique, and further optimisation is possible: for example, *minimal revenue* tolls create the desired SO flow pattern at minimal additional cost to the user.

In the case of deterministic assignment, the minimal revenue toll problem is capable of solution by various methods, such as linear programming (Bergendorff et al, 1997) and by reduction to a multi-commodity max-flow problem (Dial, 2000). However, these methods are restricted in their application to small networks or problems of special structure.

This paper develops new methodologies to examine the minimal revenue toll problem in the case of Stochastic User Equilibrium. Tolling solutions for both 'true' System Optimum and Stochastic System Optimum under SUE are derived, using both logit and probit assignment methods.

Keywords: Traffic assignment; Stochastic user equilibrium; Probit model; Logit model; Optimal tolls; Marginal social costs
Topic Area: C6 Network Design, Optimal Routing and Scheduling

1. Introduction

1.1. General background to road user charging

Road Tolling is a commonly used term, but can be used to describe different situations. For example there are many instances of 'toll roads' particularly in continental Europe, whereby a charge is made for travel along usually a section of high quality trunk road. Similarly a charge may be made to use a short length of road, primarily in the case of a tunnel or a bridge as is common in the UK. Such charges are usually either fixed or related to distance travelled, and payment is made at a toll booth at one end of the charged section, either electronically, or by actual payment at the booth.

Congestion charging by means of implementing road user tolls, has been much discussed, but has been implemented in relatively few cities. Toll rings exist and are operational in Oslo and Bergen in Norway, and area charging schemes exist in Singapore and now in London. These operational road user charging schemes have used a cordon system, which has the benefit of being transparent and easy to implement, and acts to discourage drivers from entering the controlled area, but once the driver is within the cordon, there is no additional incentive to choose a route which would be beneficial to the system as a whole.

Intuitively it would seem logical that if road tolls are to be implemented, that they should in some way be optimal; that is they should be as effective as possible with regard to specified criteria. It may be a political objective to maximise revenue, within limits of political acceptability, whilst not seeking particularly to discourage road users or to lessen congestion, which would lead to relatively cheap tolls. If instead congestion reduction were the primary objective, tolls would be set very high to discourage usage, an extreme case of which would be to completely restrict traffic and impose high fines for violation.

If optimality is desired however, suitable criteria must first be defined. Theoretically this is often considered by fixing network demand, and then considering how that traffic may be assigned throughout the network such that the overall network cost is minimised.

Whilst operational schemes are cordon based trials have however been carried out in which tolling schemes have been tested with road pricing measures such as: distance travelled, time spent travelling and congestion caused (Cambridge study (May and Milne, 2000), (Ison, 1998)), which demonstrate that the technology to implement a path or link based tolling system for urban areas does exist, and so such schemes may be feasible for actual implementation in the future.

There is also current political interest in the UK regarding more developed tolling schemes: The Commission for Integrated Transport recently published a report 'Paying for Road Use' (CfIT, 2002), which suggests the introduction of nationwide road user charging. The report is of particular interest in that it suggests the use of marginal social cost pricing on all roads (i.e. motorways, A Roads, minor roads, city centres etc), balanced by a reduction (or abolition), of Vehicle Excise Duty, combined with a reduction in fuel duty, so that the result desired would be fiscal neutrality. Such a scheme would rely on charging for travel along a link, rather than passing across a cordon, and would therefore require similar technology to implement as would be required for a minimal revenue toll scheme.

1.2. Modelling the effect of road tolls

Traffic assignment models seek to replicate the traffic pattern which is created when drivers choose their routes across a network from their origin to their destination.

In the case of deterministic assignment, it is assumed that drivers, with perfect network knowledge, will act selfishly to minimise their personal travel cost, resulting in the Wardropian User Equilibrium (UE) flow pattern. Tolls may then be imposed to ‘force’ a UE assignment to result in an alternative desired flow pattern. The Social (or System) Optimum (SO) is one such desired flow pattern, where the Total Network Travel Cost (TNTC) is minimised, and occurs when all used routes between any OD pair have equal marginal cost. The UE and SO solutions correspond to Wardrop’s first and second equilibrium principles (Wardrop, 1952). Tolls that result in such a flow pattern being created are non-unique, the classical economics solution being Marginal Social Cost Price (MSCP) tolls, whereby a toll equal to the difference between the marginal social cost (that is imposed upon the network by the driver) and marginal private cost (that the driver experiences) is levied on each link. Other constraints may however be imposed: such as toll revenue being minimised (subject to all link tolls being strictly non-negative), resulting in MinRev tolls; or fiscal neutrality being obtained, resulting in Robin Hood tolls (Hearn and Ramana, 1998), where the non-negativity condition normally imposed upon toll sets is relaxed.

The Minimal Revenue toll problem has, in the case of deterministic assignment, been solved such that the System Optimal solution is obtained, by various methods: for example, Linear Programming (Bergendorff et al, 1997) and reduction to a multi-commodity max-flow problem (Dial, 2000). However an efficient link based method that does not rely on Linear Programming solutions has yet to be developed in the mutiple origin case; a single origin algorithm however exists (Dial, 1999).

Route spreading, which is an observed phenomenon in traffic assignment, can be modelled by applying cost-flow relations to simulate congestion as in the case of deterministic assignment. Stochastic assignment methods however assume that instead of drivers having a 'perfect' knowledge of the varying OD costs of a network, they have a variable perception of these costs. Stochastic user equilibrium (SUE) assignment is based on the premise that each driver will act to minimise their perceived route cost, which follows a distribution such as those given in the logit or probit models.

Traditionally deterministic assignment has been used to model congested urban networks. If the same methods are applied though to uncongested inter-urban networks they tend to result in an All or Nothing type solution which is unrealistic in practice. Stochastic methods may be used to successfully model inter-urban networks, but it is desirable to have a single method which will be capable of modelling both extremes (and the middle ground). Thus Stochastic User Equilibrium (SUE) methods have been developed (Maher and Hughes, 1997). It would seem logical that drivers do perceive costs differently from each other, either because of different levels of network knowledge or different priorities (e.g. avoidance of right turns or roundabouts, minimising distance or time), and so the use of a stochastic method would seem to be more realistic and thus it is useful to extend the concept of tolling to the stochastic case.

This paper therefore develops new methodologies to examine the minimal revenue toll problem in the case of Stochastic User Equilibrium. A discussion of stochastic assignment methods is given in section 2.

In examining the case of Stochastic User Equilibrium the 'desired flow pattern' to be created must first be determined. The classical economics solution of replacing cost flow functions with marginal cost flow functions, does not generally result in the total network cost being minimised in the stochastic case (Yang, 1999). Thus tolls which are analogous to Marginal Social Cost Pricing (MSCP) in the deterministic case do not give the Deterministic System Optimal flow solution.

If the 'true' system optimal flow pattern is desired, it may be possible to derive tolls which are unrelated to MSCP. It is not obvious if such tolls exist or under which conditions they may exist and, if they *are* found to exist, if they are unique. If toll sets exist which are not unique, then as in the case of UE, it would be possible to impose additional constraints, and to search for (for example) minimal revenue tolls. Tolling methodologies to approach the SO solution under SUE are developed in section 3.

It may however be considered to be more desirable in the stochastic case to produce instead a 'Stochastic System Optimum' (SSO) where the *perceived* total network cost is minimised. Tolling to achieve the SSO solution is the subject of section 4.

2. Stochastic assignment models

Stochastic methods are based on the assumption that a driver minimises their perceived cost, or chooses the alternative that gives the highest utility.

Utility functions U_k may be expressed as the sum of a deterministic component V_k and a random error component ξ_k, where k is a member of the set of alternatives.

i.e.

$$U_k = V_k + \xi_k \qquad \forall k \qquad (1)$$

The probability that an alternative is chosen is the same as the probability that that alternative has highest utility in the choice set.

Whilst not being entirely exhaustive (Sheffi, 1985), the most commonly used stochastic method models assume either a Normal distribution (probit models), or the Gumbel distribution (logit models), for the drivers' perception error ξ_k .

The logit model is based on the use of the logistic function, which is a choice function used to choose between two or many alternatives.

It may be written:

$$p_i = \exp(-\theta C_i) \;/\; \sum_j \exp(-\theta C_j) \qquad (2)$$

where p_i is the probability of choosing alternative i, C_i is the costs associated with route i and θ is a dispersion parameter; the lower the value of θ, the higher the level of uncertainty, conversely a high value of θ would correspond to drivers having an accurate view of actual route costs, ie the deterministic case.

The logit formulation has the advantage of mathematical tractability, and has been used initially for that reason, but logit based loadings have a significant disadvantage in that they do not account for overlapping paths in a satisfactory manner. For example three completely distinct paths would have flow assigned in the same way as a single path together with two paths including a significant overlap. If each path had around equal cost, then each path would be assigned around one third of the traffic irrespective of any overlap. In addition the logit method assigns traffic based on an absolute difference in cost (time), for example a five minute difference in journey time will produce the same route choice proportions whether the difference relates to route times of 5 and 10 mins or route times of 200 and 205 mins. In the first case one route takes twice as long as another, whilst in the second, the five minute difference may well not be perceived as 'any difference at all'. It would seem reasonable to require a model to account for the difference in journey time in relation to the total journey time when assigning traffic.

The probit model assumes that the random error term is normally distributed, and that the joint density function of these errors is Multivariate Normal (MVN).

Thus the probability distribution of cost for each link is Normal, with mean μ being the value of the link cost flow relation, and variance σ^2 assumed to be proportional to the mean.

$$\beta = \text{Variance/Mean} = \sigma^2 / \mu \quad \Rightarrow \quad \sigma^2 = \mu\beta \qquad (3)$$

$$c_a \sim N\left(c_a(x_a)\,,\, \beta c_a(x_a)\right) \qquad \forall a \qquad (4)$$

The probit model solves the problem of overlapping paths by the use of correlations between the path cost perception errors, but does none the less have associated drawbacks.

In the case of only two alternatives, it is reasonably straightforward to calculate the probabilities for travel on each link, but this is not the case if there are more alternatives.

There are various methods of solution for the many alternative case: one is numerical integration of a multiple integral. This method had been considered to be computationally prohibitive for situations with more than about four or five alternative routes, however recent work (Rosa and Maher, 2002) shows that feasible numerical integration approaches now exist which can be used for networks with up to around twenty alternative routes. A second alternative utilises 'Clarke's Method' (Clark, 1961), where a successive approximation method is used, where the maximum of two normally distributed random variables is approximated by another Normal variable, and the Stochastic Assignment Method SAM (Maher and Hughes, 1997) is based on this. This method is applied iteratively, but does create a heuristic solution.

Alternatively a Monte Carlo simulation may be used, whereby a random value representing the perceived travel time of a link, is sampled from the density function for that link, and an All or Nothing assignment is carried out based on the set of sampled perceived travel times across all network links. The process of sampling and assignment is repeated (multiple times) and averaged to give the final flow pattern. The major drawbacks of Monte Carlo probit methods, are due to the issue of repeatability and to long computation times. For example if this method were used to compare different highway scheme proposals, it may be unclear how much of a projected improvement may be due to random effects. In practice the method may have to be carried out a very large number of times to obtain a sufficient level of accuracy and stability, and would therefore be costly from a computational perspective.

3. System optimal road tolls

3.1. Path-based methodology

If it is desired that an SUE assignment using the original cost/flow functions with the addition of a toll, should produce the SO flow pattern that is obtained under deterministic assignment, where the TNTC is minimised, then using logit-based SUE this may be formulated as below:

$$X_i = 1000\exp\text{-}\theta(C_i + T_i) \,/\, \sum_j \exp\text{-}\theta(C_j + T_j) \qquad (5)$$

where C_i and X_i are the path costs and flows at the SO solution which may be found using deterministic assignment methods, and T_i are the desired path tolls to be determined.

The 'toll difference' between pairs of path tolls for each OD pair may then be found by the division of pairs of equations, thus;

$$T_j - T_i = (1/\theta)\ln(X_i/X_j) + (C_i - C_j) \qquad (6)$$

A resulting order of magnitude of path tolls (for each OD pair) may be deduced, and assuming tolls to be non-negative, and seeking minimal revenue tolls, the smallest toll path (for each OD pair) may be set as zero, and the remaining path tolls calculated.

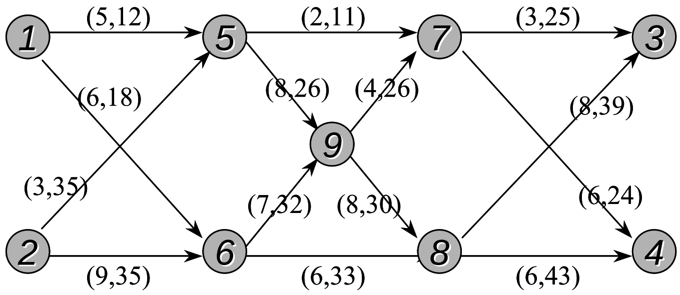
However the use of the logistic function to determine path differences, requires all path flows to be non-zero, as zero flows would clearly result in infinite tolls. Thus it is only theoretically possible to derive toll sets that will make a Stochastic User Equilibrium assignment method produce the Wardropian SO flow pattern in those networks where all paths do have non-zero flow at SO. There are consequently difficulties to be encountered when dealing with more complex networks, (Smith et al, 1994) where generally there will exist technically feasible paths which have zero flow at SO.

Thus it is necessary in general to derive toll sets which produce an approximate SO solution rather than a precise one.

In the case of Stochastic assignment, some flow, however small, will theoretically be allocated to every feasible path between OD pairs.

As the variability parameter θ tends to infinity, the stochastic assignment tends to deterministic, thus if marginal cost flow functions are used the stochastic assignment will tend towards the SO flow pattern as θ increases. Each iteration of such a stochastic assignment would assign non-zero flows to every path (although they may still be small), and so the problem of zero flow paths could be overcome by using a stochastic 'MSCP' assignment with large value of θ as the desired flow pattern rather than the exact SO.
The toll difference equation (6) could then be used to derive path tolls.

This method is illustrated using the 9-node network with 2 origins and 2 destinations as shown in Figure 1 below. This network has been frequently used in the literature (Bergendorff et al (1997), Dial (2000); a restricted version is used here (with 4 vertical links (5↔6, 7↔8) carrying zero flow removed), to render the network a-cyclic (as for Dial's multiple origin algorithm), and thus limit the path enumeration matrix so that 24 viable paths are obtained (six paths between each of the four OD pairs).



OD Pair: [1,3] [1,4] [2,3] [2,4]
Demand: 10 20 30 40
$c_a(x_a) = c_a^{(0)}(1 + 0.15(x_a/Y_a)^4)$

Figure 1 – Nine-node network diagram, showing OD demand and link cost flow relations

The link cost functions are of BPR type as shown where ($c_a^{(0)}$,Y_a) for each link are given on the diagram, x_a is link flow, c_a is link cost, $c_a^{(0)}$ is free flow link cost and Y_a is link capacity.

For the above network, the minimum TNTC = 2253.9

All links have non-zero flow at the Wardropian SO solution but this is not true of all paths: in a Wardropian SO assignment using 30 iterations of the Method of Successive Averages (Sheffi, 1985), 4 paths were completely unused.

Therefore a desired path flow set was obtained using stochastic assignment methods for $\theta = 15$; for which the TNTC = 2254.0 , which is very close to the 'true' SO.

A viable toll set is given below for $\theta = 0.1$.

OD pair	[1,3]						[1,4]						[2,3]						[2,4]					
T =	12	9	56	0	39	2	15	16	18	16	12	0	11	1	45	5	39	0	16	11	7	28	19	0

The path tolls correspond to the paths given in the path-link incidence matrix A (Appendix1), with 6 paths for each OD pair.

Thus it would appear achievable to create viable path toll sets, that will create a flow pattern approaching the true SO flow pattern, as closely as is desired under logit SUE. However in the limiting case ($\theta \rightarrow \infty$), some tolls will tend to infinity, which is clearly not desirable, and so an appropriate degree of closeness to the SO solution would need to be determined.

In addition it may be possible to exclude paths with very small flows from the tolling calculations, as if such paths are of such high cost that they are never used in a true SO solution, they may be excluded from a set of 'sensible paths', which may be used instead of the full set of feasible paths. Path based assignment methods which are being developed (Rosa, 2001), are based on partial (efficient) path enumeration techniques.

A further difficulty with this method is that although a viable path toll set can be determined, it is not possible to derive a consistent link based toll set from it. The inconsistency may be demonstrated by combining sets of paths as is given in Figure 2 below.

Paths 2 and 5 together contain the same links as paths 3 and 4 together, therefore for consistent link tolls the total toll on paths 2 and 5 (total=48) should equal the total toll on paths 3 and 4 (total=56). This is clearly not the case (similar examples of inconsistency may also be demonstrated), so consistent link tolls may not be determined.

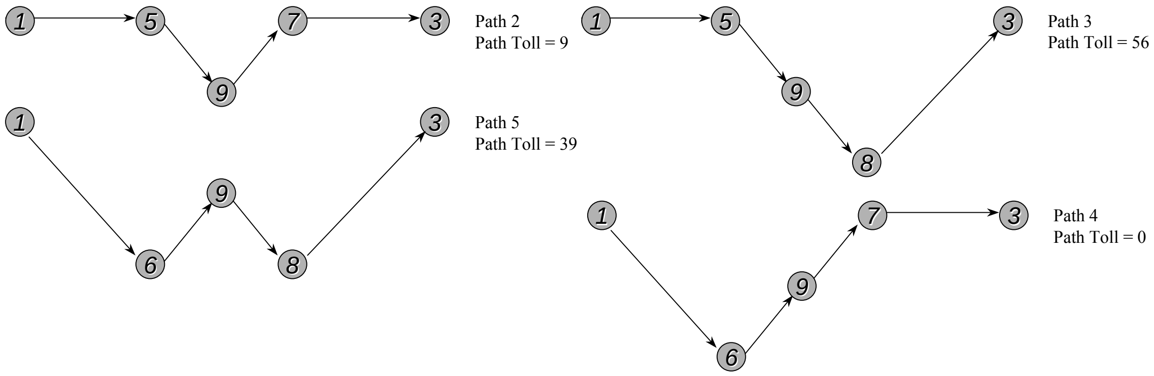


Figure 2 – Path combinations from Bergendorff's nine-node network

Thus it must be considered whether a path based tolling methodology would be sensible for implementation, even if developing efficient path based assignment methods could be utilised. A tolling solution where the same link has a different cost depending on the overall route travelled, would intuitively not appear to be equitable or practicable. Further, the technology required to implement a path based tolling scheme, would require vehicle tracking, which although technically feasible would result in concern over privacy issues.

Further, the path difference equations used require the use of logit assignment, which does not model networks with overlapping paths as well as probit. If a network with only two links is used, it is possible to solve for tolls algebraically in the probit case, but this is not the case for any more complex network.

Consequently this method is not considered to be of potential use for practical implementation.

3.2. Link-based methodology

A link-based methodology to derive tolls that would create a flow pattern approaching the SO is therefore desirable. It was assumed from the previous results, that link-based tolls might not be sufficient to replicate the desired SO flow pattern in the limiting case, but that good sub-optimality would be acceptable for practical purposes.

The objective is still to minimise the total network travel cost, and this is attempted by seeking a link flow pattern that approaches the flows obtained under deterministic SO assignment. Thus links where the flow is higher than that desired have link costs progressively increased by the addition of a toll until the desired flow pattern is approached, as in the procedure given below:

Step 1: Deterministic SO Assignment: $\mathbf{F}_{\mathrm{SO}}$, $\mathbf{C}_{\mathrm{SO}}$ and $\mathrm{TNTC}_{\mathrm{SO}}$ obtained

Step 2: Link toll vector set to zero: $\mathbf{T}_0 = \mathbf{0}$

Step 3: Set $n = 0$

Step 4: SUE assignment: $\mathbf{C}_n$ and $\mathbf{F}_n$ obtained

Step 5: Calculate: $\mathbf{P} = (\mathbf{F}_n - \mathbf{F}_{\mathrm{SO}})(\left|\mathbf{C}_n - \mathbf{C}_{\mathrm{SO}}\right|)$

Step 6: Determine link j where $P(j)$ is greatest.

Step 7: *Perform iteration to calculate $t(j)$ s.t $F(j) = F_{SO}(j)$ to required degree of accuracy.

Step 8: $\mathbf{T}_{n+1} = \mathbf{T}_n + \mathbf{t}$; where $t(i) = t(j)$ when $i{=}j$ and $t(i) = 0$ otherwise

Step 9: Calculate TNTC: Stop if TNTC sufficiently close to $TNTC_{SO}$
or set $n = n + 1$ and repeat from Step 4.

*Step 7 requires an internal iteration, where only the output flow for the link in question is regarded. The initial value for t on this link, is set at the cost difference as per Step 5, the output flow difference for that link with that toll is then examined, and the link toll re-estimated using linear interpolation. The internal iterative procedure is repeated until the link flow is sufficiently close to the desired SO flow for that link.

This method is illustrated using the previously used 9-node network (see Figure 1), with logit SUE where $\theta = 0.1$.

In Step 1 a deterministic SO assignment determines the desired link flow set, and the link costs are calculated for these flows. The minimum value of the TNTC is also recorded (for this example $TNTC_{SO} = 2253.9$).

An initial toll vector is then set with all tolls being zero (Step 2).

An SUE assignment is then completed, and the link costs, link flows and the TNTC compared with those desired.

The toll set is constructed in a step-wise process, where only a single link toll is considered in each iteration; thus the link to be tolled in that iteration must be chosen.

Steps 4 and 5 determine which link is chosen: choosing simply the link where the flow was most in excess of the desired SO flow for that link would not take into account the relative costs, and so a product of flow and cost difference is used here, although this may be refined in future work. As only non-negative tolls are being imposed, the absolute value of the cost difference is used, so that the chosen link, where the value of the product is greatest has a flow strictly greater than that desired.

Table 1 below shows the stepwise construction of a toll set.

Table 1 – Iterative building of 'Optimising' toll set

Iteration	0	1	2	3	4	5	6	7	8	9	10	11	12
t_1 (1-5)	-	-	-	-	-	-	-	-	-	-	**0.9**	0.9	0.9
t_2 (5-7)	-	**7.2**	7.2	7.2	7.2	7.2	7.2	**8**	8	**8.9**	8.9	8.9	8.9
t_3 (7-3)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_4 (1-6)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_5 (2-5)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_6 (5-9)	-	-	-	-	-	-	-	-	**0.6**	0.6	0.6	0.6	0.6
t_7 (9-7)	-	-	-	-	-	-	**1.4**	1.4	1.4	1.4	1.4	1.4	1.4
t_8 (6-9)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_9 (9-8)	-	-	-	**13**	13	13	13	13	13	13	13	**13.8**	13.8
t_{10} (7-4)	-		**7.9**	7.9	**12.9**	12.9	12.9	12.9	12.9	12.9	12.9	12.9	12.9
t_{11} (8-3)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_{12} (2-6)	-	-	-	-	-	**3.6**	3.6	3.6	3.6	3.6	3.6	3.6	**3.8**
t_{13} (6-8)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_{14} (8-4)	-	-	-	-	-	-	-	-	-	-	-	-	-
TNTC	2441	2385	2337	2285	2268	2262	2259	2258	2257	2256	2255	2255	2254
REV	0	154	307	449	568	705	746	759	777	797	813	819	822

It can be seen from Table 1 and from the graph in Figure 3 below, that the first few iterations are by far the most significant, and no great benefit is gained from continuing to approach the $TNTC_{SO}$ for many iterations. Further if it is desirable to keep as many links toll free as is possible, it is not then sensible to continue to add small tolls on additional links, to reduce the TNTC only by tiny amounts.

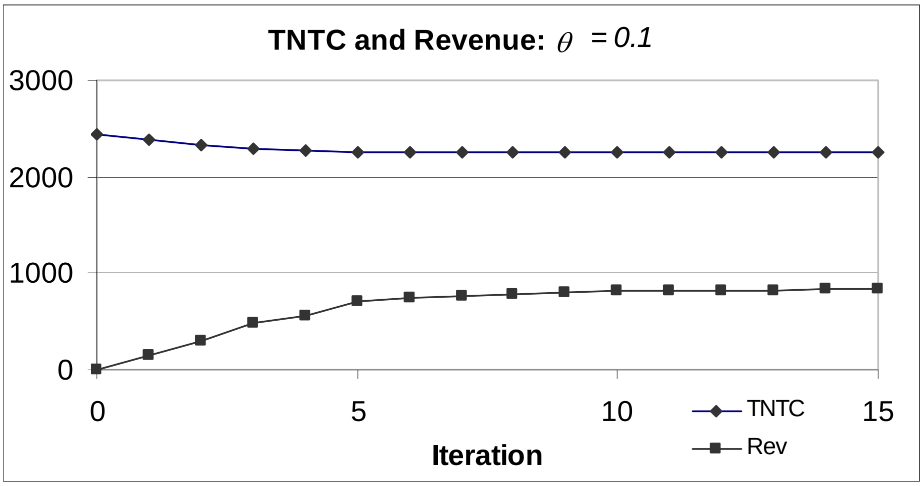


Figure 3 – Total Toll Revenue required for reduction in TNTC

The link toll set resulting from the 12 iterations given above, is shown in Figure 4 below, where link width is proportional to the size of the link toll. The TNTC achieved after 12 iterations is only 0.02% greater than the Minimum TNTC. However if the process was stopped after only 4 iterations, the TNTC achieved is still only 0.6% greater than $TNTC_{SO}$ and 4 links that could be tolled, would remain toll-free.

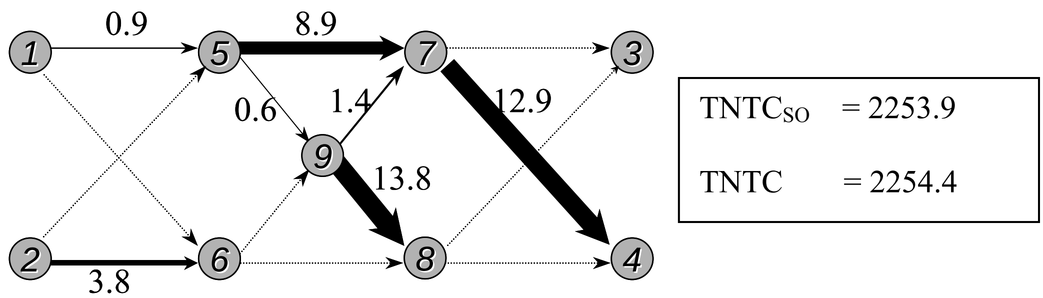


Figure 4 – logit toll-set for Bergendorff's network (θ = 0.1) – 12 iterations

Whilst this methodology has been demonstrated using logit assignment, it may equally be used for other stochastic assignment models. Figure 5 below, shows the reduction in TNTC achieved for different values of the dispersion parameter θ using logit assignment, and for $\beta = 0.5$ using probit assignment (the Stochastic Assignment Method SAM (Maher and Hughes, 1997)).

It must be noted that the method used does not result in the TNTC strictly decreasing at every iteration, although the overall trend is that it does reduce as the desired flow pattern is approached. The internal iteration at Step 7, has in these examples been used to reduce the flow on a particular link so that it is very close to the desired flow value for that link at SO. During this internal iteration process, at some point the value of the difference product P will be greatest for a new link, after this point, the overall TNTC may no longer decrease. It is possible to amend this internal iteration, so that the link toll is determined at the minimal value for the TNTC that can be achieved by just varying the toll on this link. However it appears in practice, that as this will generally give a smaller toll being added at each iteration, that it causes a greater number of the main iterations to be required. Consequently, the objective at each internal iteration is that the flow difference on that link should be reduced to (approximately) zero.

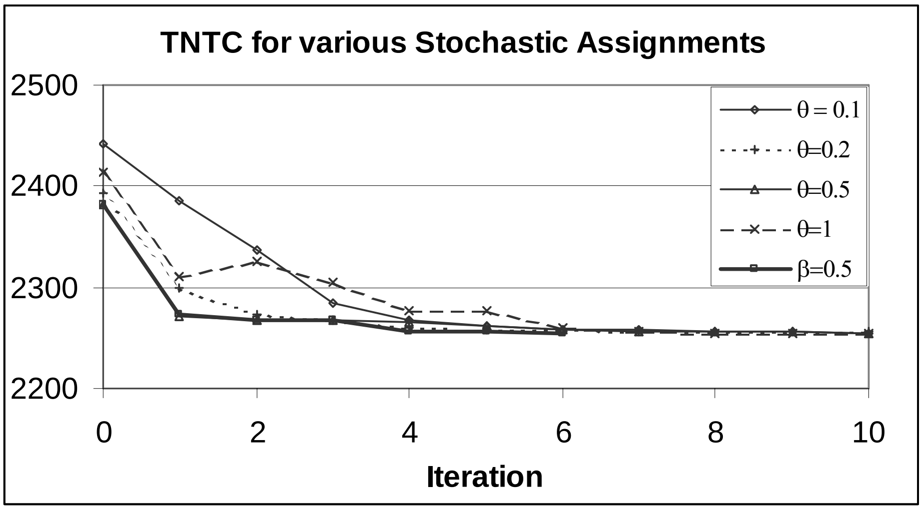


Figure 5 – TNTC with increasing iterations for various Stochastic Assignments

It of interest to note that if the logit and probit models are to be compared using the relation:

$$\text{Var}\,(U_k) = \pi^2/(6\theta^2) \qquad (7)$$

(Cascetta, 1990), where U_k is the probit utility function as in equation (1), then despite the link variances in the probit case obviously being different for each link (as each link has a different free flow cost value), an approximate correspondence can be found in this case between probit $\beta = 0.5$, and a logit sensitivity parameter of $\theta \approx 0.5$. It can be observed in Figure 5 above, that the graphs for $\beta = 0.5$ and $\theta = 0.5$ predominantly coincide.

4. Stochastic social optimum road tolls

In the case of stochastic user equilibrium, it could be argued that it is not the 'actual' or deterministic total travel cost which should be minimised, but rather the perceived total network travel cost.

In the case of deterministic assignment, it is well known that that the Total Network Travel Cost is minimised and the System Optimal flow pattern is obtained, when cost flow functions are replaced by marginal cost flow functions. Recent work (Maher and Stewart 2004), has shown that the analogous case is true under stochastic assignment.

A Stochastic Social Optimum (SSO) was formulated such that at the SSO solution, the total of the users' perceived costs is minimised. It was shown that under a general utility-maximising framework that includes the two most important cases of logit and probit loading, the augmented path costs at the SSO solution were the marginal social costs, and hence the relationship of the SSO solution to the SUE is the same as that of SO to UE. In particular, the SSO solution can be found by means of an SUE algorithm, by replacing the standard path costs by the marginal social path costs in the stochastic loading; and the same form of toll set that is optimal in the deterministic case is optimal in the stochastic case.

Thus MSCP tolls may be easily found using existing link-based assignment methods.

The minimal revenue toll problem, is thus similar to that in the deterministic case, differing though in that all used paths will clearly not have a common cost, and so existing techniques may not be utilised directly.

If path enumeration is feasible, as it is for the 9-node network used previously, then Min-Rev toll sets may be determined from the MSCP toll sets. If a link-toll set exists (ie MSCP), the path-enumeration matrix may be used to obtain the corresponding path-toll set. A zero-toll tree, may then be derived, such that the path with the smallest MSCP toll for each OD-pair is given a zero-toll, and the other paths have their toll reduced by the same amount.

The zero-toll trees thus obtained, depend on the value of the sensitivity parameter in both logit and probit cases.

The remaining link-tolls may be derived from the Min-Rev path toll set by matrix algebra, using a reduced path-enumeration matrix, where the zero-toll paths are removed.

4.1. An illustrative example

Logistic assignment has again been used in this example, but the method used is equally applicable to other stochastic models.

The zero-toll trees for Bergendorff's 9-node network (Figure 1), are shown in Figure 6 below; the zero-toll links being represented by the bold print arrows. As θ increases the driver's assumed perceived knowledge of network costs increases, so that as θ tends to infinity, the logit stochastic assignment tends towards a deterministic assignment, and the final zero-toll tree (θ=5) is indeed the same as that obtained by deterministic methods.

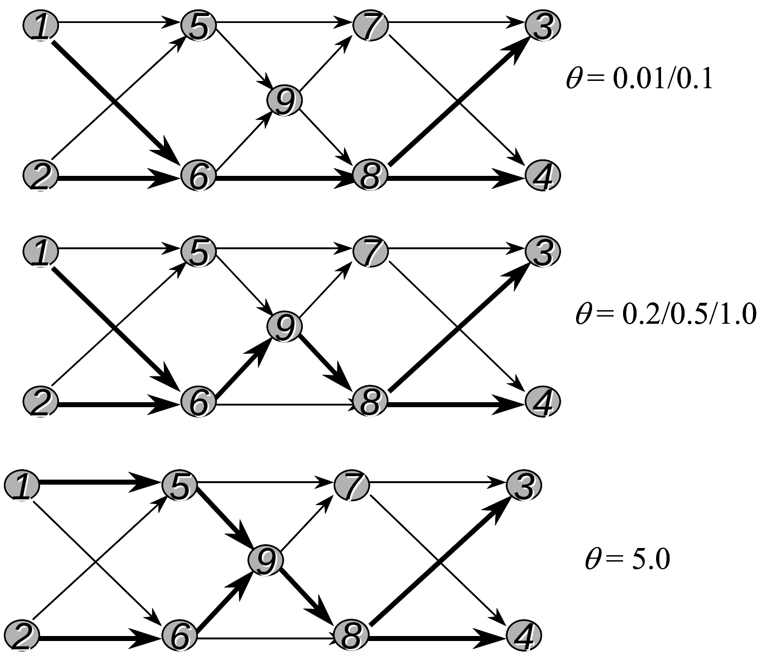


Figure 6 – Zero-toll trees as θ varies

In determining minimal revenue toll sets, if there are a greater number of viable paths than there are links (as will generally be the case), the minimal revenue path-toll vector (which is easily determined), together with the path enumeration matrix will result in an over-determined system, which is why a viable path-toll set will not necessarily yield consistent link-tolls as demonstrated in section 2.1. However in this the SSO case, the system of equations in fact reduces to one which is underdetermined, and so various equally optimal link-toll solutions exist, and further optimisation may be possible if desired. In particular it is of interest to obtain as many links with zero-toll as possible, but even with this provision, in this example there were four equally optimal toll sets for each value of θ.

A possible Min-Rev toll-set is given below in Table 2 for various θ.

The links corresponding to the zero-flow paths are highlighted. Other zero-toll links may be observed, although it must be remembered that there are other equally optimal solutions which are not shown. Despite the existence of three distinct zero-toll trees for varying values of the sensitivity parameter, the change in individual link-toll values as θ varies appears to be reasonably smooth.

Table 2 – Minimal Revenue Toll Sets as θ varies

Toll \ θ	0.01	0.1	0.2	0.5	1	5	Deterministic
t1 (1-5)	12.8	2.9	1.0	0.2	0.1	0.0	0.0
t2 (5-7)	0.0	5.4	6.9	7.7	7.9	8.0	8
t3 (7-3)	0.1	2.6	3.7	5.1	5.9	6.9	7.2
t4 (1-6)	0.0	0.0	0.0	0.0	0.0	0.0	0
t5 (2-5)	6.0	2.4	2.2	2.9	3.4	3.9	4
t6 (5-9)	5.3	0.0	0.0	0.0	0.0	0.0	0.0
t7 (9-7)	0.0	0.0	0.0	0.0	0.0	0.0	0
t8 (6-9)	9.1	0.7	0.0	0.0	0.0	0.0	0.0
t9 (9-8)	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t10 (7-4)	6.4	3.9	3.3	3.1	3.1	3.2	3.2
t11 (8-3)	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t12 (2-6)	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t13 (6-8)	0.0	0.0	1.9	4.4	5.6	6.8	7.2
t14 (8-4)	0.0	0.0	0.0	0.0	0.0	0.0	0

5. Summary

It has generally been seen as desirable to solve network traffic problems by link based methods to avoid path enumeration, and its associated high computational costs. The continuing advance in computing capabilities however, has made path based computational methods more feasible than previously, and research is currently developing path based methods for traffic assignment.

In the case of road user charging, it is relevant to consider whether a path based tolling system would be either sensible or implementable. A link based system would be more obviously possible, by use of a smart card within the vehicle, and roadside beacons on the tolled links. A path based system however, would require the vehicle to be tracked throughout the network, and the appropriate toll for the complete route chosen to be charged at the point of exit; this may be feasible with a GPS on-vehicle navigational system, but such a system would need to be installed into all vehicles, so this would not be a viable solution in the immediate future. In addition it should be considered whether a path based system would be seen to be 'fair' in that it would essentially charge different prices for travel along a certain link, depending on which overall path were chosen. A viable tolling scheme would need to be politically acceptable.

In attempting to approach the ‘true’ SO flow pattern through tolling, the algebraic logit formulation derived path-tolls which could not then be separated into consistent link tolls.

Also the issue of small path flows encountered in the logit case would result in unreasonably large tolls on some routes. Further, algebraic methods are untenable in the probit case, and so such methods were not felt to be desirable, and instead an iterative heuristic link based method has been derived.

For the toy-network used here for illustration the desired SO flow pattern where TNTC was minimised could be closely approached, within a small number of iterations. This method does however require extension to examine larger networks to see how close to the TNTC it is possible to get in general. A sensible trade off between the cost imposed upon the drivers to achieve the reduction in TNTC, and the actual reduction obtained would need to be established for practical purposes. In additional it may be desirable to require certain links to be zero-tolled, and this could be included in this type of process.

In attempting to achieve the Stochastic Social Optimum flow pattern, by use of minimal-revenue tolling, the marginal social cost price tolls, known to create the desired flows, were used as a starting point. Path enumeration was then required to use these to derive minimal revenue path-based tolls and from these, link-based tolls. The minimal-revenue toll problem in this case is analogous to that for deterministic assignment, but with the stochastic nature of the assignment causing all used paths not to have a common cost.

It would be possible here to use an iterative method similar to that used in seeking to approach the 'true' SO, but if possible it would be more desirable to utilise the easily established MSCP tolls as a starting point, but to derive a fully link based procedure. This is an area of ongoing work.

The desired flow pattern to be achieved in the stochastic case remains though an issue to be resolved. Is it more desirable in the stochastic case to minimise 'real' or 'perceived' costs throughout a network?

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Appendix1

A =

1	1	1	0	0	0	0	0	0	0	0	0	0	0	[1,3]
1	0	1	0	0	1	1	0	0	0	0	0	0	0	
1	0	0	0	0	1	0	0	1	0	1	0	0	0	
0	0	1	1	0	0	1	1	0	0	0	0	0	0	
0	0	0	1	0	0	0	1	1	0	1	0	0	0	
0	0	0	1	0	0	0	0	0	0	1	0	1	0	
1	1	0	0	0	0	0	0	0	1	0	0	0	0	[1,4]
1	0	0	0	0	1	1	0	0	1	0	0	0	0	
1	0	0	0	0	1	0	0	1	0	0	0	0	1	
0	0	0	1	0	0	1	1	0	1	0	0	0	0	
0	0	0	1	0	0	0	1	1	0	0	0	0	1	
0	0	0	1	0	0	0	0	0	0	0	0	1	0	
0	0	0	0	1	0	0	0	0	0	0	0	0	0	[2,3]
0	0	1	0	0	0	1	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	1	1	0	1	1	0	0	
0	0	0	0	0	0	0	0	0	0	1	1	1	0	
0	1	0	0	1	0	0	0	0	1	0	0	0	0	
0	0	0	0	1	1	1	0	0	1	0	0	0	0	
0	0	0	0	1	1	0	0	1	0	0	0	0	1	[2,4]
0	0	0	0	0	0	1	1	0	1	0	1	0	0	
0	0	0	0	0	0	0	1	1	0	0	1	0	1	
0	0	0	0	0	0	0	0	0	0	0	1	1	1	