# DEVELOPMENT OF A DELAY EQUATION VALID FOR MOVEMENTS WITH SHORT LANES AND BASED ON THE WEBSTER'S MODEL FOR TRAFFIC LIGHTS

*Gonçalo Santos, Department of Civil Engineering and Architecture, Instituto Superior Técnico, Technical University of Lisbon*

*Álvaro Seco, Department of Civil Engineering, Science and Technology Faculty, University of Coimbra*

*José Neves, Department of Civil Engineering and Architecture, Instituto Superior Técnico, Technical University of Lisbon*

## ABSTRACT

The presence of a short lane at an intersection regulated by traffic lights directly influences the movement's saturation flow. It can be verified that the saturation flow falls off during the green time period when the last vehicle stored in the short lane crosses the stop line. This variation in saturation flow is not considered in Webster's delay equation developed in 1958. To overpass this issue, some authors tried to arrange simple adaptations to the design process, based on the referred equation. However, these methodologies could differ from the values that minimize the intersection delay, because Webster's expression is not valid for the specific problem of movements with short lanes.

This paper describes the development of an expression that resulted from the adaptation of Webster's delay model to movements with short lanes. It starts by explaining the important steps of the work of Webster concerning to the delay equation and then focuses on the adaptation of the deterministic and stochastic terms of his expression.

The result is a new expression, valid for under-saturated conditions, that allows the estimation of delay for movements with short lanes. It can be verified that Webster's equation is a particular case of this new expression.

*Topic for submission:* Traffic Theory and Modelling.

*Keywords:* traffic lights, short lane, delay, Webster.

# 1. INTRODUCTION

In 1958, Webster published an important paper with the results conducted at the Road Research Laboratory (UK), into the delays at fixed-time traffic signals and into its optimum settings (Webster, 1958). From his work resulted a semi-empirical expression to estimate the average delay per vehicle using a single approach to an intersection controlled by fixed time traffic signals, for under-saturated conditions ($x$<1).

$$d = \frac{C\left(1-\frac{g}{C}\right)^2}{2\left(1-y\right)} + \frac{x^2}{2q(1-x)} - 0,65\left(\frac{C}{q^2}\right)^{1/3} x^{(2+5\lambda)} \qquad (1.1)$$

*d – average delay per vehicle on the approach;*
*C – cycle time;*
*λ – proportion of the cycle that the signal is effectively green (g/C);*
*q – arrival flow;*
*s – saturation flow;*
*x – degree of saturation, ratio between average number of arrivals per cycle and the maximum number of departures per cycle (x=q/λs).*

The Webster's expression is based on two models: an uniform arrival based delay model that estimates the value of the deterministic parcel of delay (first parcel of equation) and an random arrival based delay model that estimates the value of the stochastic delay (second parcel of equation). The third parcel of the equation is a calibration factor that allows the adjustment of the theoretical values to the ones obtained by simulation.

The models used by Webster to develop expression (1.1) consider that, during the effective green time, the vehicles are discharged at a constant rate, denominated by saturation flow. This premise compromises its application to situations where the saturation flow falls down suddenly to a lower rate, which is the case of the existence of a short lane at the approach. Consequently, Webster's equation for the optimum cycle, that results from the two first terms of expression (1.1) (see Webster, 1958), has the same limitation. For that, Webster's optimum cycle equation cannot be used to optimize intersections with movements including short lanes.

The scope of this paper is to define a delay equation valid for movements with short lanes, based on the adaptation of the two theoretical models of Webster's equation. These are sufficient to define a methodology that estimates the optimum signal settings, which is the next result of the research carried out.

# 2. DEFINITION AND CHARACTERISTICS OF A SHORT LANE

## 2.1 Definition

A short lane is a geometric element that consists in a lane with limited length, which is insufficient to feed the discharging of vehicles during effective green time, causing an instantaneously decrease of the movement's saturation flow. The reduction of saturation flow, denominated by short lane effect, occurs when the last vehicle stored in the short lane crosses the stop line.

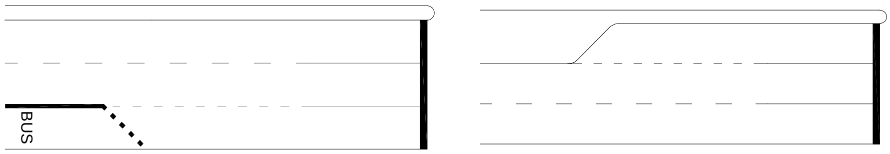


**Figure 1: Representation of short lanes (design cases)**

During the design of an intersection, namely at places where insertion space is very limited, short lanes can be used to increase the number of vehicles that potentially can depart during the green light signal (see Figure 1). But there are other situations where the short lane effect can occur due to temporarily defective service conditions, such as road works or illegally parking (see Figure 2).

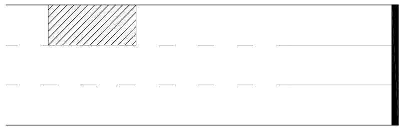

**Figure 2: Representation of a short lane originated by the presence of an obstacle**

Limited length lanes can also be used to segregate different movements that initially were associated to the same lane (for example, going forward and turning left, see Figure 3). This case will not be considered in this work, because segregated movements have their own demand and saturation flow and consequently are analysed in a different way.

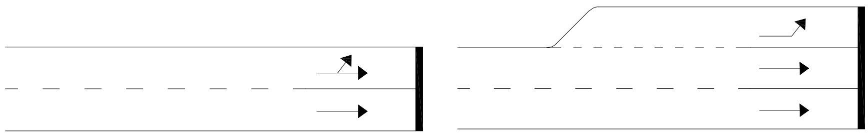

**Figure 3: Separation of movements using a limited length lane**

## 2.2 Design elements

A short lane is defined by three design elements: length, green time and red time.

The length defines physically the short lane and its value ($L_{sh}$) corresponds to the distance between the point where the lane is large enough to accommodate a vehicle and the interior limit of the stopping line. The value $L_{sh}$ can be converted in number of vehicles using the expression (2.1), where $j$ is the average length that a vehicle occupies in a traffic line. $N_{sh}$ is denominated by storage capacity of the short lane and is useful to keep expressions uncomplicated.

$$N_{sh} = \frac{L_{sh}}{j} \tag{2.1}$$

The green time is related to the number of vehicles that potentially can depart in each cycle ($N_{\max}$). For a normal lane, that is a lane without length limitations, the value of $N_{\max}$ is given by expression (2.2).

$$N_{\max} = g \times s \tag{2.2}$$

For a short lane, the value of $N_{\max}$ has a upper limit that corresponds to the number of vehicles that physically can be stored inside the short lane (see expression (2.3)).

$$N_{\max} = \min\left[(g \times s); N_{sh}\right] \tag{2.3}$$

For $N_{\max}$ to be equal to $N_{sh}$, it is required that the green time has the exact value to discharge all the vehicles that the short lane can store, at the rhythm of the saturation flow. This value is calculated by expression (2.4). In other words, the short lane contributes to the flow of vehicles until $g=g'$.

$$g' = \frac{N_{sh}}{s_{sh}} \tag{2.4}$$

*$g'$ – green time during which the short lane is contributing to the flow of vehicles;*
*$s_{sh}$ – saturation flow associated to the short lane (equal to the saturation flow of a normal lane with the same characteristics of the short lane).*

The red time allows the accumulation of vehicles at the approach. To maximize the use of the short lane, it is important that the red time given to the movement is sufficient to let vehicles occupy the entirely storage capacity of the short lane. This happens when the red time value is equal or greater than the minimum value given by expression (2.5).

$$r_{\min} = \frac{N_{sh} \times n}{q} \tag{2.5}$$

*$r_{\min}$ – minimum red time;*
*$q$ – arrival flow (demand);*
*$n$ – number of lanes of the considered movement.*

## 2.3 Saturation flow of a movement including a short lane

The behaviour of a short lane can be illustrated considering two sets of lanes (example adapted from Akcelik (1981)): set (1) includes the short lane with length $L_{sh}$; and set (2) includes the remaining lanes, which lengths are big enough to not be considered as short lanes (see Figure 4). Admit that all vehicles move forward and that the demand at the approach allows the existence of a continuous flow during the discharging of vehicles.

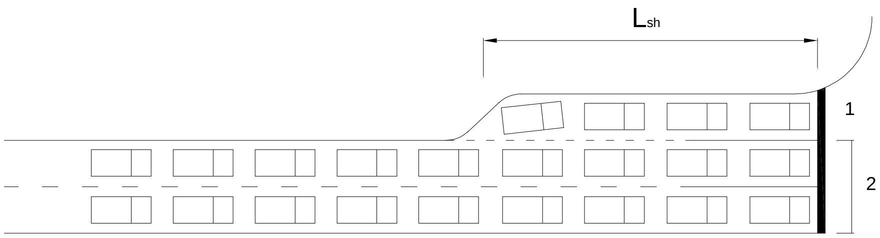


**Figure 4: Example to explain the behaviour of a short lane**

During the green time period $g < g'$, all lanes are contributing to the flow of vehicles that cross the stopping line. For this case the saturation flow of the movement is equal to the sum of the saturation flows of sets 1 and 2. This value is designated by $s_{\max}$.

At the end of this period, the storage capacity of the short lane is empty and the saturation flow assumes the value of the remaining lanes (set 2), designated by $s_{\min}$. This value is verified until the end of the green period.

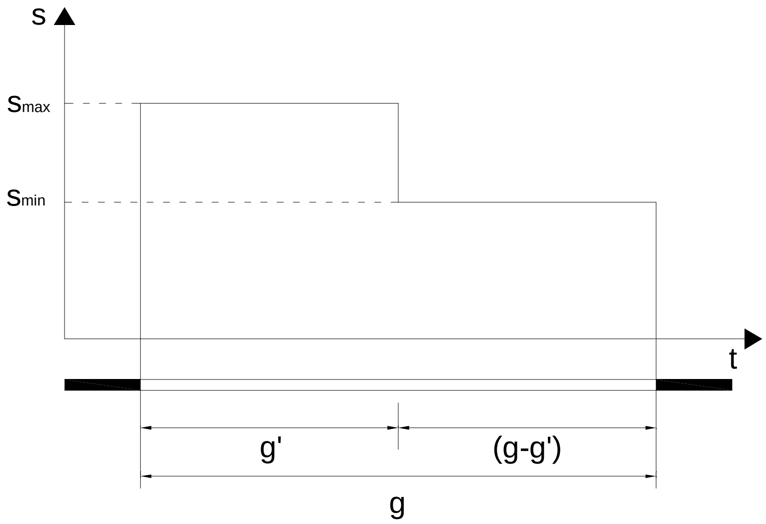


**Figure 5: Variation of the saturation flow during green time**

Considering the behaviour described before, the value of the saturation flow of a movement including a short lane is based on the existence or inexistence of an instantaneously reduction of the flow of vehicles during green time period. Therefore, to determine the saturation flow of a short lane, it is important to know if there is short lane effect. This verification can be made based on the model that characterizes the variation of the saturation flow during the green time period (see Figure 5) and using expression (2.4):

If $g \le g'$, there isn't short lane effect; the saturation flow is determined using the same procedure as for normal lanes;

If $g > g'$, there is short lane effect; the saturation flow of the movement ($s_{avg}$) depends on the value of green time and is calculated by a weighted average of the minimum and maximum saturation flows, taking in account its durations.

$$s_{avg} = s_{\max}\frac{g'}{g} + s_{\min}\left(1 - \frac{g'}{g}\right) = \frac{N_{sh}}{g} + s_{\min} \qquad (2.6)$$

# 3. THE EQUATION OF WEBSTER

To develop a new expression is necessary to look at the work of Webster and try to understand the foundations of the models that are behind his equation. As referred above the equation of Webster has three terms (see expression (1.1)): the first term is based on a model that considers vehicles arriving at a uniform rate, the second term models the random arrival of vehicles, and the third term gives an empirical correction of the values.

The third term was established by a regression analysis using the differences between the delay values obtained by simulating traffic behavior at approaches controlled by traffic lights (with different values of $C$, $q$, $g/C$ and $x$) and the ones estimated by the first two terms of the delay expression (Allsop, 1972).

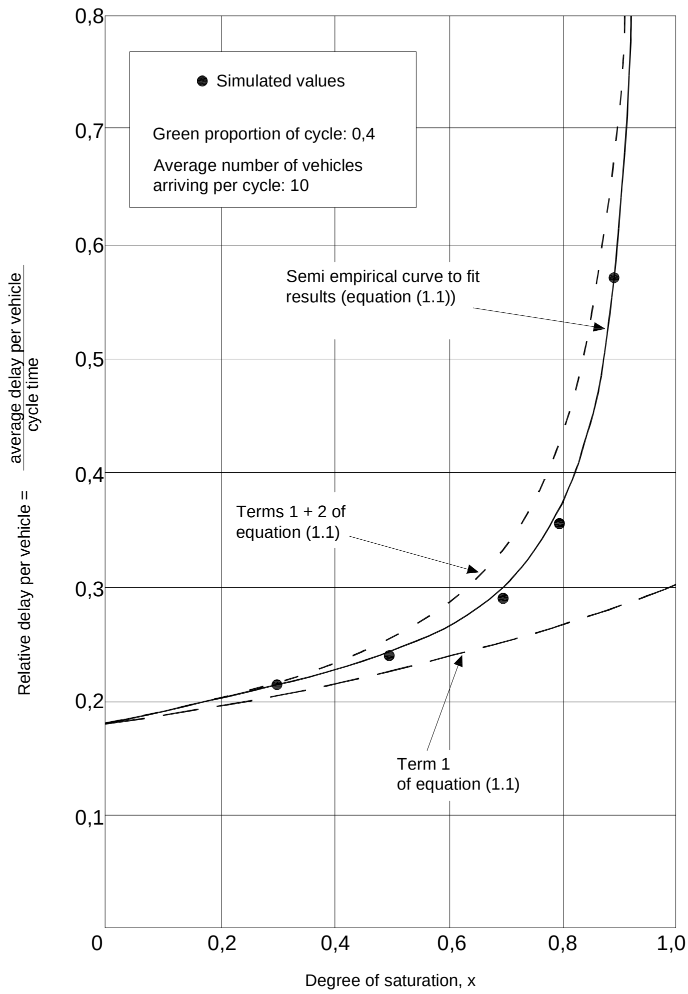


**Figure 6: Typical fixed time delay curve (Webster, 1958)**

The values simulated by Webster and the curves for the delays estimated using the first term, the first and second terms, and equation (1.1), are represented in Figure 6. It can be seen that the two first terms of the expression (1.1) have a fair agreement with the simulated delays for all levels of flow. The addition of the third term improves slightly the agreement of the curve, giving a close fit to the simulated values. Webster stated that, the delay for most practical purposes can be calculated by expression (3.1), since the empirical term represents 5 to 15 per cent of the value given by expression (1.1) (Webster, 1958).

$$d = \frac{9}{10}\left[\frac{C\left(1-\frac{g}{C}\right)^2}{2(1-y)} + \frac{x^2}{2q(1-x)}\right] \quad (3.1)$$

As a consequence, Webster considered only the first two terms of expression (1.1) to perform the minimisation of the total delay at an intersection, which originated the optimum cycle equation used to obtain the optimal signal settings. A detailed analysis of the models behind the two first terms of Webster's delay equation is presented bellow.

## 3.1 Uniform arrivals model

The first term (deterministic) considers that vehicles are arriving at a uniform rate. For this situation the delay is estimated based on the variation of the cumulated vehicles during time. The functions related to the cumulated number of vehicles that arrive with a flow *q* during the red light and the cumulated number of vehicles that depart at a rhythm defined by the saturation flow (*s*) during the effective green time (*g*), are represented In Figure 7.

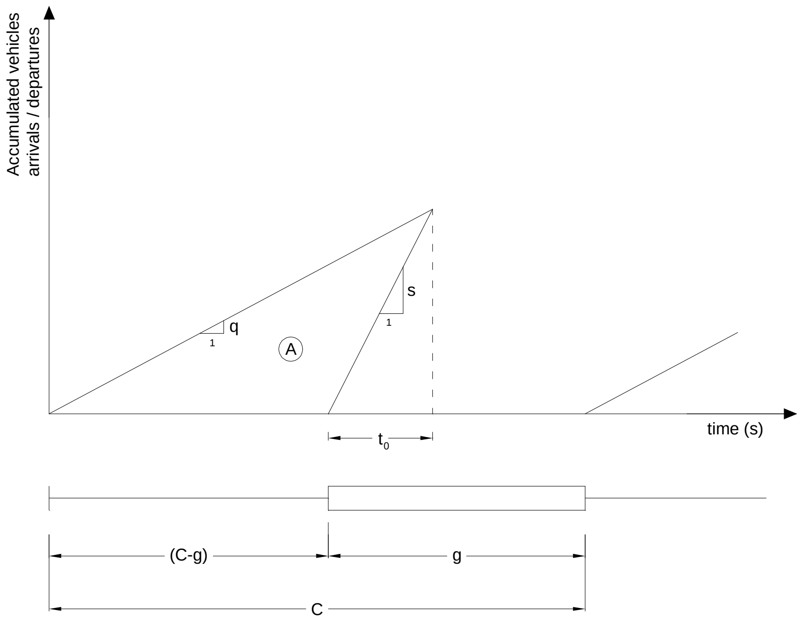


**Figure 7: Variation of the accumulated number of vehicles**

The expression (3.2) can be defined from the representation of the model.

$$q \times \left[(C-g)+t_0\right] = s \times t_0 \quad (3.2)$$

Separating $t_0$ from the other variables,

$$t_0 = \frac{q \times (C-g)}{s-q} \quad (3.3)$$

The cumulated delay per cycle is obtained from the area identified by A.

$$D = A = \frac{[(C-g)+t_0] \times st_0}{2} - \frac{s \times t_0^{\,2}}{2} = \frac{s \times t_0}{2}(C-g) \tag{3.4}$$

Replacing $t_0$ by expression (3.3),

$$D = \frac{sq(C-g)^2}{2(s-q)} \tag{3.5}$$

The average delay per vehicle ($d$) is calculated by dividing the cumulated delay by the number of vehicles that arrive during the cycle.

$$d = \frac{D}{q \times C} = \frac{C\left(1-\frac{g}{C}\right)^2}{2(1-y)} \tag{3.6}$$

The expression (3.6) is the first term of Webster's delay equation.

## 3.2 Random arrivals model

The second term of the delay expression introduces the random nature of arrivals, considering the possibility of saturation problems during some cycles, characterized by the formation of queues.

According to Allsop (1972), Webster, by suggestion of Welding, considered the existence of a queuing system between the traffic that arrives at the intersection and the traffic lights (see Figure 8), in a way that the number of vehicles arriving next to the signal during each cycle is equal to the number of vehicles that depart during green time. A description for what happens can be read on Webster (1958): "*It is an expression for the delay experienced by vehicles arriving randomly in time at a 'bottleneck', queuing up, and leaving at constant intervals*".

Summing up, the model uses a queue system at upstream of the traffic lights where the vehicles arriving randomly are stored, forming a queue. The server of the queue releases those vehicles at a constant time interval and the number of vehicles released during each cycle is equal to the number of vehicles that can depart during green time.

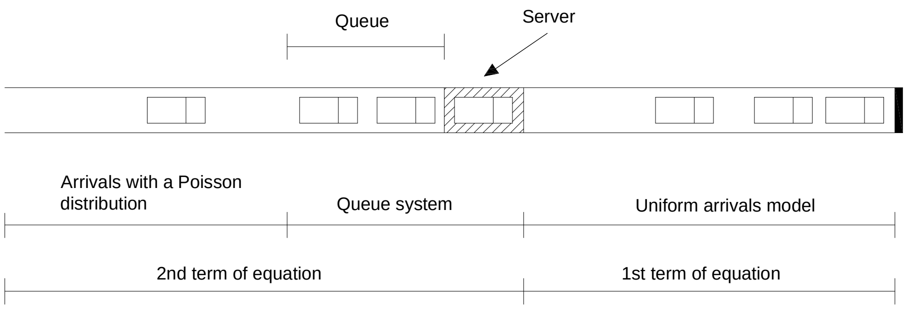


**Figure 8: Representation of the delay model (1st and 2nd terms)**

The waiting queue system defined is M/D/1 type, according to Kendall notation. This corresponds to having a Poisson distribution for the arrival of vehicles, a fixed service time and one server.

The value of the service time is established based on the premise previously presented, referring that the number of vehicles served by the queue system during each cycle has to be equal to the number of vehicles that depart during the effective green time.

This way,

$$t_s = \frac{C}{sg} \tag{3.7}$$

*$t_s$ – service time;*
*$C$ – cycle time;*
*$s$ – saturation flow;*
*$g$ – effective green time.*

The equation to calculate the average value of delay for this type of queue system was presented by Kendall in 1951, based on the Pollaczek-Khintchine formula (Allsop, 1972).

The system is analyzed by dividing time in certain instants, denominated by regeneration points, corresponding to the end of service, that is, the moment immediately before the departure of vehicles (Kendall, 1951). Remember that service time is deterministic and the time interval between regeneration points is equal to $t_s$ seconds. Considering two consecutive regeneration points $t$ and $t+t_s$, it is possible to establish a relation between the number of vehicles in the system at those two time instants (see Figure 9 and expression (3.8)).

$$n' = \max\left(n-1;0\right) + p \tag{3.8}$$

*$n$ – number of vehicles in the system at the regeneration point $t$ ;*
*$n'$ – number of vehicles in the system at the regeneration point $t + t_s$ ;*
*$p$ – number of vehicles that arrive to the system during the time interval $[t ; t + t_s]$.*

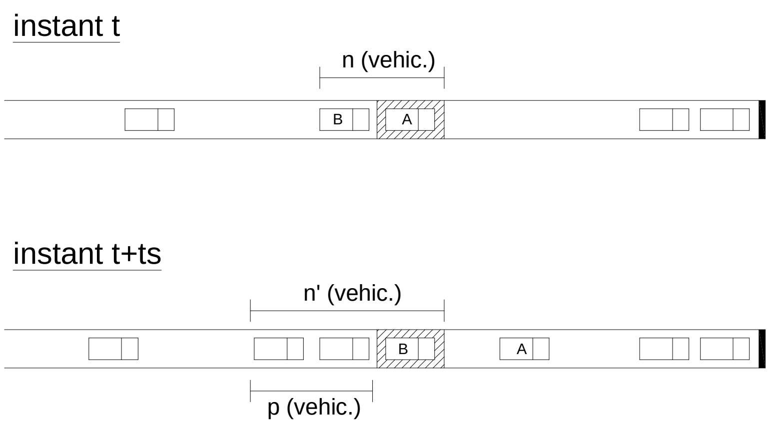


**Figure 9: Representation of the consecutive regeneration points**

The expression (3.8) can be rewritten using a factor $\delta$ related to $n$, to guarantee that $n'$ never assume a negative value.

$$n' = n - 1 + \delta + p$$
$$\delta = \begin{cases} 1 \text{ se } n = 0 \\ 0 \text{ se } n > 0 \end{cases} \qquad (3.9)$$

Taking in account the definition of the function $\delta$, it can be established the following relations:

$$\begin{aligned} \delta^2 &= \delta \\ \delta n &= 0 \end{aligned} \qquad (3.10)$$

From the model represented by expression (3.9) and Figure 9, is possible to characterize the variable $p$, which is related to the arrival of vehicles and follows a Poisson distribution.

The average number of vehicles that arrive to the system between points of regeneration, $E(p)$, is equal to:

$$E(p) = q \times t_s = \frac{qC}{sg} \qquad (3.11)$$

The variable $q$ represents the flow of arrivals.

The relation between the number of vehicles that arrive during one cycle ($q$×$C$) and the number of vehicles that depart during the green time at the saturation flow rhythm ($s$×$g$) is denominated by saturation degree ($x$). Having in consideration that, in a Poisson distribution, the variance is equal to the average, it comes:

$$E(p) = Var(p) = x \qquad (3.12)$$

The value of $E(p^2)$ is obtained from the Konig theorem ($Var(x) = E(x^2) - [E(x)]^2$).

$$E(p^2) = Var(p) + [E(p)]^2 = x + x^2 \qquad (3.13)$$

The expression (3.9) can be rewritten having in account that the relation between the random variables continues valid if the mean values are used.

$$E(n') = E(n) - 1 + E(\delta) + E(p) \qquad (3.14)$$

Assuming the existence of statistical equilibrium, the random variables that represent the number of vehicles in the system have the same distribution.

$$\begin{aligned} E(n') &= E(n) \\ E(n'^2) &= E(n^2) \end{aligned} \qquad (3.15)$$

The expression (3.14) can be simplified.

$$E(\delta) = 1 - E(p) = 1 - x \tag{3.16}$$

Squaring both members of the expression (3.9) and using the expressions presented at (3.10), it results:

$$\begin{aligned} & n'^2 = \left(n - 1 + \delta + p\right)^2 \\ \Leftrightarrow\ & n'^2 = n^2 - 2n\left(1 - p\right) + (p-1)^2 + \delta\left(2p - 1\right) \end{aligned} \tag{3.17}$$

The variables $n$ and $p$ are independent, that is, the probability of arrival between two consecutive regeneration points is independent from the length of the waiting queue. The variables $\delta$ and $p$ are also independent because $\delta$ depends only on $n$.

Considering the expected values of the random variables,

$$E(n'^2) = E(n^2) - 2E(n)\left(1 - E(p)\right) + (E(p)-1)^2 + E(\delta)\left(2E(p) - 1\right) \tag{3.18}$$

Assuming again a statistical equilibrium,

$$\begin{aligned} & 0 = -2E(n)\left(1 - E(p)\right) + (E(p)-1)^2 + E(\delta)\left(2E(p) - 1\right) \\ \Leftrightarrow\ & E(n) = \frac{E(p)^2 - 2E(p) + 1 + E(\delta)\left(2E(p) - 1\right)}{2\left(1 - E(p)\right)} \end{aligned} \tag{3.19}$$

Considering the expressions (3.12), (3.13) e (3.16),

$$E(n) = \frac{(x + x^2) - 2x + 1 + (1 - x)\left(2x - 1\right)}{2\left(1 - x\right)} \tag{3.20}$$

Simplifying expression (3.20),

$$E(n) = \frac{2x - x^2}{2\left(1 - x\right)} \tag{3.21}$$

*E(n)* represents the average number of vehicles inside the system at each regeneration point.

Considering the known relation of queuing theory,

$$L = \lambda \times W \tag{3.22}$$

*L – average number of elements in the system;*
*λ – arrival rate (elements/time);*
*W – system average waiting time.*

It results,

$$E(n) = q \times W \tag{3.23}$$

Changing $E(n)$ by expression (3.21),

$$W = \frac{2x - x^2}{2q(1-x)} \tag{3.24}$$

Finally, knowing that the average waiting time of the system ($W$) is equal to the average waiting time in queue ($W_q$) plus service time ($t_s$), it can be obtained the expression that determines the value of the average queue waiting time per vehicle ($d$).

$$\begin{aligned} & W = W_q + t_s \\ & \Leftrightarrow W_q = \frac{2x - x^2}{2q(1-x)} - \frac{C}{sg} = \frac{2x - x^2}{2q(1-x)} - \frac{x}{q} \end{aligned}$$

$$d = W_q = \frac{x^2}{2q(1-x)} \tag{3.25}$$

The expression (3.25) corresponds to the second term of the Webster's delay equation.

# 4. DELAY EQUATION FOR APPROACHES INCLUDING SHORT LANES

To estimate the delays correctly for an approach with a short lane, it is necessary to adapt the theoretical models of Webster's expression.

## 4.1 Adaptation of the uniform arrivals model

The deterministic term of Webster's delay equation is based on a model that considers a constant saturation flow during green time. For approaches with short lanes, the saturation flow depends on green time, cycle extension and storage capacity of the short lane. This will change the potential departure flow, originating three different situations:

**Situation A:** $g' < g \wedge t_x \leq g'$

The cycle time is not enough to empty the storage capacity of the short lane. In this case, the potential departure flow is considered equal to the maximum saturation flow, in other words, there isn't short lane effect (see Figure 10).

As stated before, the value $g'$, represents the parcel of green time during which the short lane is contributing to the departure rate of vehicles. Its value is calculated by expression (2.4) using $N$ instead of $N_{sh}$ (see expression (4.1)), since the number of vehicles that effectively are expected to use the short lane can be different from the number of vehicles that potentially can be stored.

$$g' = \frac{N}{s_{sh}} \tag{4.1}$$

*N - number of vehicles that are expected to use the short lane (theoretically this value can be considered equal to $N_{sh}$).*

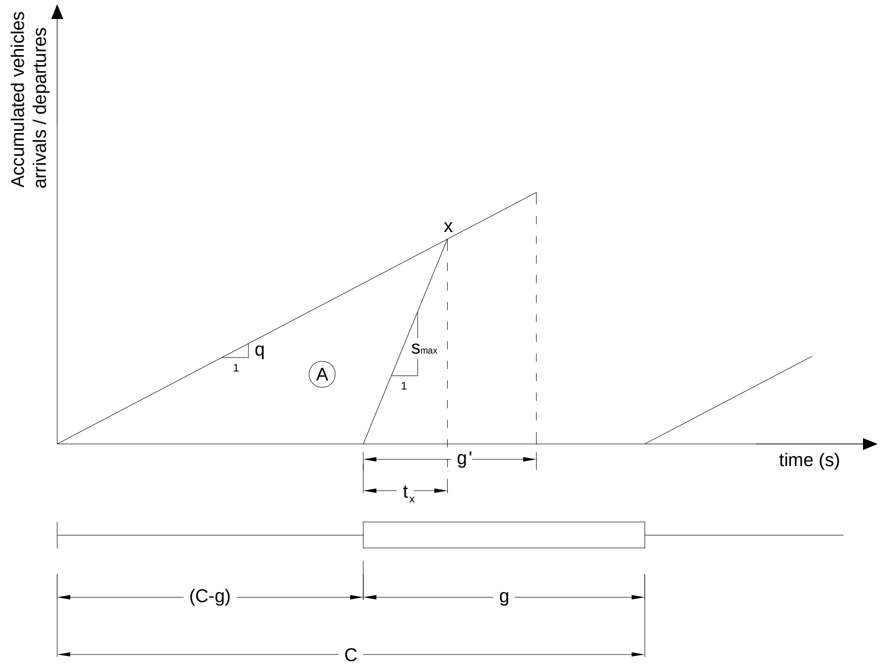


**Figure 10: Variation of the accumulated number of vehicles for a movement with short lane ($t_x$<$g'$<$g$)**

The value $t_x$ is obtained from the intersection between the lines with slope $q$ and $s_{max}$ (designated by $x$).

$$q\left(C-g+t_x\right) = -(C-g)s_{\max} + s_{\max}\left(C-g+t_x\right) \tag{4.2}$$

Putting $t_x$ at the left side of the equation,

$$t_x = \frac{q\left(C-g\right)}{s_{\max}-q} \tag{4.3}$$

It can be concluded that this model is similar to the one developed by Webster, therefore the expression that determines the accumulated delay per cycle is obtained from expression (3.5) changing the variable $s$ by $s_{\max}$.

$$D = \frac{s_{\max} q (C-g)^2}{2(s_{\max} - q)} \tag{4.4}$$

And the delay per vehicle is equal to

$$d = \frac{D}{q \times C} = \frac{s_{\max}(C-g)^2}{2C(s_{\max} - q)} \tag{4.5}$$

This expression is valid for situations that respect the two conditions bellow.

$$g' < g \Rightarrow N < s_{sh} \times g \tag{4.6}$$

$$t_x \le g' \Rightarrow \frac{q(C-g)}{s_{\max} - q} \le \frac{N}{s_{sh}} \Rightarrow N \ge \frac{q\, s_{sh}(C-g)}{s_{\max} - q} \tag{4.7}$$

Joining these two conditions, it can be defined a dominium of applicability:

$$\frac{q\, s_{sh}(C-g)}{s_{\max} - q} \le N < s_{sh} \times g \tag{4.8}$$

**Situation B:** $g' < g \wedge t_x > g'$

The cycle time allows the growing of the waiting lines beyond the limit of the short lane. Consequently, it is verified an instantaneously decreasing of the saturation flow, from $s_{\max}$ to $s_{\min}$. The referred variation of the saturation flow is called short lane effect and occurs at the instant when the green time given to the movement is equal to *g'* (see Figure 11).

From the model represented by Figure 11, it can be verified that:

$$q\left[(C-g) + t_0\right] = s_{\max} \times g' + s_{\min}(t_0 - g') \tag{4.9}$$

Working on the expression (4.9) it is possible to obtain $t_0$ in function of the other variables.

$$t_0 = \frac{q(C-g) - N}{(s_{\min} - q)} \tag{4.10}$$

(Note: as expected, the value of expression (4.9) when $t_0$=g' is equal to the value given by expression (4.2) when $t_x$=*g'*)

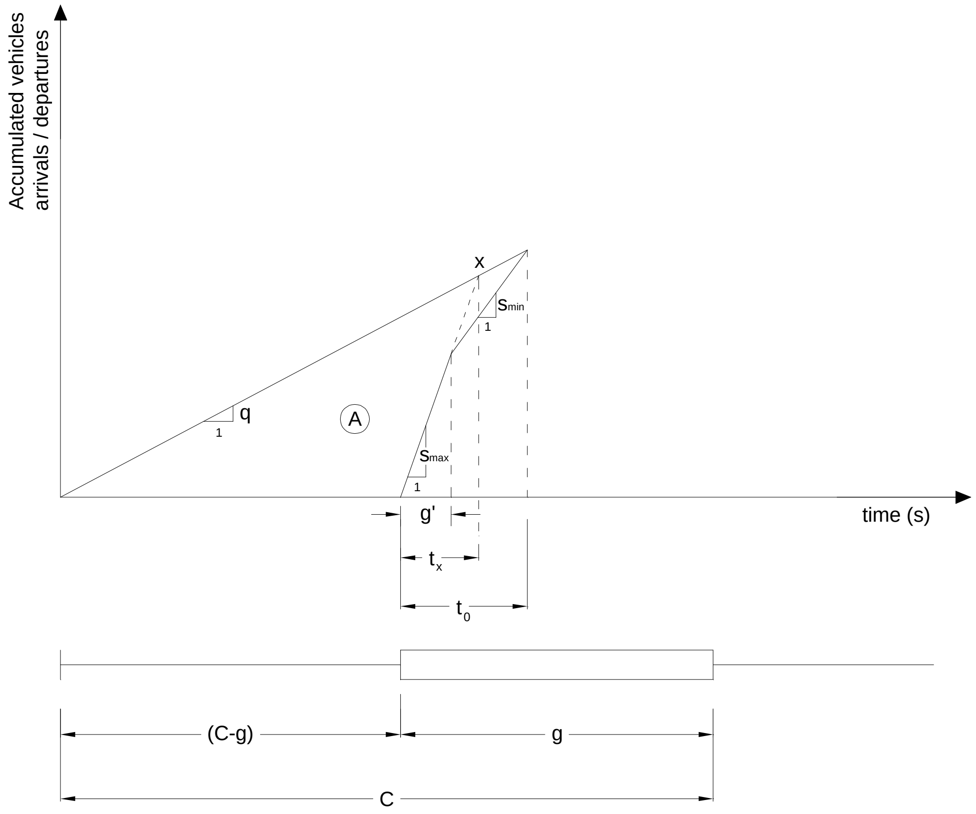


**Figure 11: Variation of the accumulated number of vehicles for a movement with short lane ($g' < t_0 < g$)**

The value of vehicle's accumulated delay per cycle for a movement including short lane is calculated by the area represented by A (inside the lines that represent arrivals, departure flows and the time axle).

$$D = \frac{1}{2}\left[ s_{sh}\left( (C-g)\,g' + (g')^2 \right) + t_0\left( (C-g)\,s_{\min} - s_{sh}\,g' \right) \right] \tag{4.11}$$

Replacing $t_0$ by expression (4.10):

$$D = \frac{1}{2}\left[ N\left( C - g + \frac{N}{s_{sh}} \right) + \frac{q(C-g) - N}{s_{\min} - q}\left( (C-g)\,s_{\min} - N \right) \right] \tag{4.12}$$

The average delay per vehicle ($d$) is obtained dividing the accumulated value of delay by the number of vehicles that arrive during the cycle.

$$d = \frac{D}{q \times C} = \frac{1}{2qC}\left[ N\left( C - g + \frac{N}{s_{sh}} \right) + \frac{q(C-g) - N}{s_{\min} - q}\left( (C-g)\,s_{\min} - N \right) \right] \tag{4.13}$$

The resulting expression is valid for $g' < g \wedge t_x > g'$. Separating the two parts of the condition it can be obtained the following relations:

$$g' < g \Rightarrow N < s_{sh} \times g \tag{4.14}$$

$$t_x > g' \Rightarrow \frac{q(C-g)}{s_{\max}-q} > \frac{N}{s_{sh}} \Rightarrow N < \frac{q\, s_{sh}(C-g)}{s_{\max}-q} \tag{4.15}$$

The condition of the expression (4.13) can be represented by the following dominium,

$$N < \min\left[\frac{q\, s_{sh}(C-g)}{s_{\max}-q}; s_{sh} \times g\right] \tag{4.16}$$

**Situation C:** $g' \geq g$

For $g' > g$, there will not be short lane effect because green time value is insufficient to allow the departure of all vehicles stored in the short lane. This case is represented by the model on Figure 12.

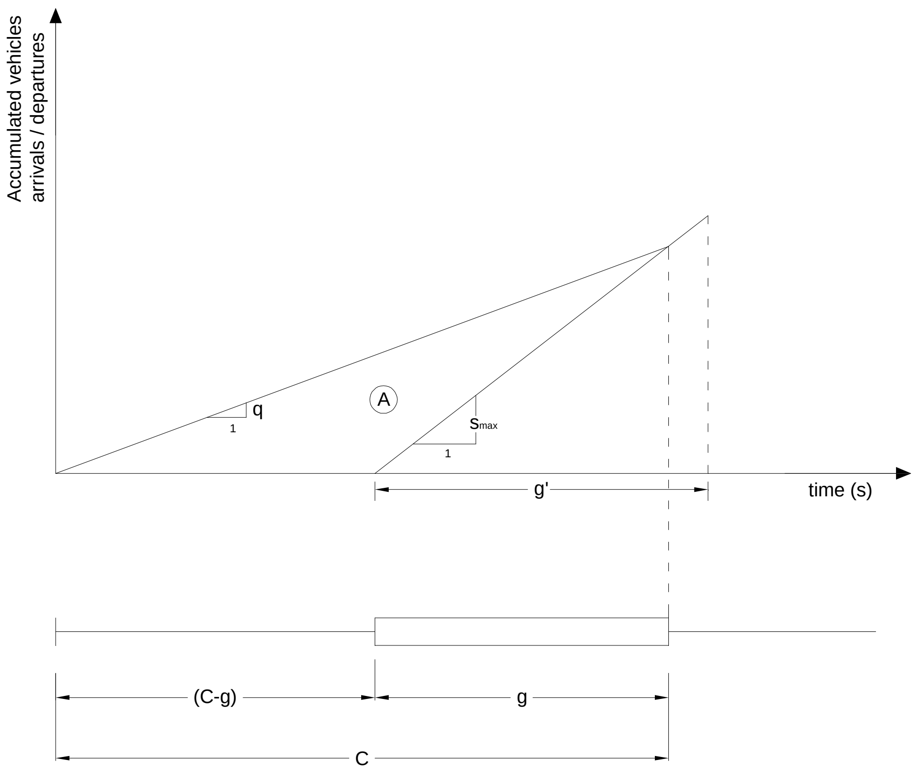


**Figure 12: Variation of the accumulated number of vehicles for a movement with short lane ($g' \geq g$)**

Once again, this model is similar to the one presented by Webster. This expression can be obtained from (3.4) replacing $t_0$ by $g$ and $s$ by $s_{\max}$.

$$D = A = \frac{[(C-g)+g]\times s_{\max} g}{2} - \frac{s_{\max}\times g^2}{2} = \frac{s_{\max}\times g}{2}(C-g) \tag{4.17}$$

The average delay per vehicle is equal to

$$d = \frac{D}{q\times C} = \frac{s_{\max}\times g}{2qC}(C-g) \tag{4.18}$$

It is possible to simplify the expression (4.18) taking in account that, for this situation, the relation $s_{\max} g = qc$ is valid.

$$d = \frac{D}{q\times C} = \frac{C-g}{2} \tag{4.19}$$

In conclusion, expression (4.19) is a particular case of expression (4.5) when $s_{\max} g = qc$ and is valid for $N \geq s_{sh} \times g$ .

The dominium of the expressions presented for the three considered situations, are separated by two limits - see Figure 13. This graphic was obtained for a representative scenario ($q_1$=900vph, $s_{sh1}$=1800vph; $s_{\min 1}$=1800vph; $N_1$=5; $q_2$=600vph; $s_2$=1800vph; $L$=10s).

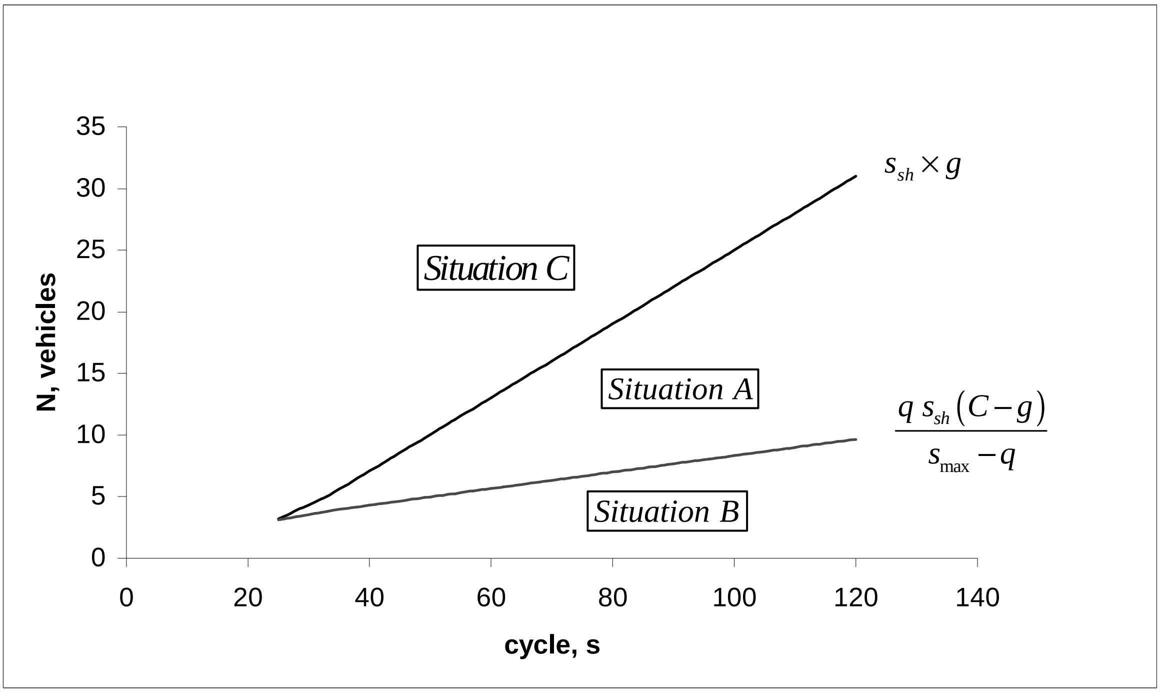


**Figure 13: Dominium of the deterministic parcel of delay for movements including short lane**

Considering Figure 13 and what was said before, it is only necessary to consider the limit between situations A and B and its expressions (4.13) and (4.5). Consequently, the deterministic parcel to calculate the average delay per vehicle at movements controlled by traffic lights including short lane is:

$$d = \begin{cases} \dfrac{1}{2qC}\left[ N\left( C - g + \dfrac{N}{s_{vc}} \right) + \dfrac{q(C-g)-N}{s_{\min} - q}\left( (C-g)s_{\min} - N \right) \right] \text{ if } N < N_0 \\ \\ \dfrac{s_{\max}(C-g)^2}{2C(s_{\max} - q)} \text{ if } N \geq N_0 \end{cases} \tag{4.20}$$

and

$$N_0 = \frac{q\ s_{sh}(C-g)}{s_{\max} - q}$$

## 4.2 Adaptation of the random arrivals model

The determination of the stochastic term of the delay for movements that include a short lane, can be made directly by expression (3.25), since the queue model used by Webster, is located between the arrivals at the intersection and the traffic lights, in a way that is not affected by what is happening close to the stopping line. There is only the need to assure that the value of the saturation degree is calculated using the correct saturation flow of the movement.

Since the saturation flow of a movement with short lane depends on the existence or not of a decrease on the saturation flow, from $s_{\max}$ to $s_{\min}$ during green time, it is possible to define two different situations:

a) $g' > g$, the maximum saturation flow lasts until the end of green time and therefore its value ($s_{\max}$) must be adopted for the determination of the delay stochastic parcel;

b) $g' < g$, the storage capacity of the short lane ends during the departure of vehicles. The value of the saturation flow depends on the value of green time and is calculated by expression (2.6) using $N$ instead of $N_{sh}$ for the same reasons presented before.

$$s_{avg} = \frac{N}{g} + s_{\min} \tag{4.21}$$

Therefore, for cases where there is short lane effect, the stochastic parcel of delay is determined by expression (4.22) using the value of $s_{avg}$ calculated by expression (4.21);

$$d = \frac{\left(\dfrac{qC}{s_{avg}\, g}\right)^2}{2q\left(1 - \dfrac{qC}{s_{avg}\, g}\right)} \tag{4.22}$$

And for the cases where there isn't short lane effect, the stochastic parcel of delay is determined by expression (4.23).

$$d = \frac{\left(\dfrac{qC}{s_{\max}\, g}\right)^2}{2q\left(1 - \dfrac{qC}{s_{\max}\, g}\right)} \tag{4.23}$$

## 4.3 Expression to determine the delay per vehicle for an approach including short lane

With the information gathered is now possible to write the equation to determine the average delay per vehicle (deterministic and stochastic parcel), for a movement with short lane, controlled by traffic lights.

$$d = \begin{cases} \dfrac{1}{2qC}\left[N\left(C - g + \dfrac{N}{s_{vc}}\right) + \dfrac{q(C-g) - N}{s_{\min} - q}\left((C-g)\, s_{\min} - N\right)\right] + \dfrac{\left(\dfrac{qC}{s_{avg}\, g}\right)^2}{2q\left(1 - \dfrac{qC}{s_{avg}\, g}\right)} & \text{if } N \le N_0 \\[2ex] \dfrac{s_{\max}\left(C-g\right)^2}{2C\left(s_{\max} - q\right)} + \dfrac{\left(\dfrac{qC}{s_{\max}\, g}\right)^2}{2q\left(1 - \dfrac{qC}{s_{\max}\, g}\right)} & \text{if } N > N_0 \end{cases} \tag{4.24}$$

and

$$N_0 = \frac{s_{sh}\; q\left(C - g\right)}{(s_{\max} - q)}$$

$$s_{avg} = \frac{N}{g} + s_{\min}$$

This expression is valid for movements controlled by traffic lights, in general (with or without short lane), and can be used as a starting point to optimize the cycle of an intersection controlled by traffic lights, following the option of ignoring the third term, like Webster did.

Notice that expression (1.1) is a particular case of expression (4.24) and can be obtained by doing the following: $N$ =0 and $s_{\min} = s_{\max} = s_{avg} = s$.

## AKNOWLEDGEMENTS

The authors would like to acknowledge the funding from TRANSPOR – Fundo para o Desenvolvimento do Ensino Avançado e Investigação em Sistemas de Transportes.

## BIBLIOGRAPHY

Akcelik, R. (1981). Traffic Signals: capacity and timing analysis. Australian Road Research Board, Vermont South, Vic.

Allsop, R. (1972). Delay at fixed time traffic signal-I: theoretical analysis, Transportation Science Vol. 6, pp. 260-285.

Catling, I. (1977). A Time-Dependent Approach to Junction Delays. Traffic Engineering and Control, 18 (11), pp. 520-523,526.

Costa, A. (1987). Cruzamentos Regulados por Sinais Luminosos. PhD thesis. Faculdade de Engenharia, Universidade do Porto.

Kendall, D. G. (1951). Some Problems in the Theory of Queues. Journal of the Royal Statistical Society, Series B (methodological), Vol. 13, No. 2., pp. 151-185.

Pereira, A. e Castilho, A. (1987). Modelo Matemático de Webster para Sinais Luminosos a Tempos Fixos. National Laboratory of Civil Engineering (LNEC), Lisbon.

Santos, G. (2008). Análise da Problemática das Vias Curtas em Modelos de Optimização de Intersecções Semaforizadas. MSc thesis. Instituto Superior Técnico, Technical University of Lisbon.

Webster, F. (1958). Traffic Signal Settings. HMSO, London.

Webster, F. e Cobbe, B. (1966). Traffic Signals. HMSO, London.