AN ANT COLONY OPTIMISATION (ACO) ALGORITHM FOR SOLVING THE LOCAL OPTIMISATION OF SIGNAL SETTINGS (LOSS) PROBLEM ON REAL-SCALE NETWORKS

D'ACIERNO, Luca; Department of Transportation Engineering, 'Federico II' University of Naples, Italy, luca.dacierno@unina.it

GALLO, Mariano; Department of Engineering, University of Sannio, Benevento, Italy, gallo@unisannio.it

MONTELLA, Bruno; Department of Transportation Engineering, 'Federico II' University of Naples, Italy, bruno.montella@unina.it

ABSTRACT

In this paper we propose an Ant Colony Optimisation (ACO) algorithm for optimising the signal settings on urban networks following a local approach. This problem, also known as LOSS (Local Optimisation of Signal Settings), has been widely studied in the literature and can be formulated as an asymmetric assignment problem (Cascetta et al., 2006). The problem consists in optimising the signal settings of each intersection of an urban network as a function only of traffic flows at the accesses to the same intersection, taking account of the effects of signal settings on costs and on user route choices.
The proposed ACO algorithm is based on two kinds of behaviour of artificial ants which allow the LOSS problem to be solved: traditional behaviour based on the response to pheromones for simulating user route choice, and innovative behaviour based on the pressure of an ant stream for solving the signal setting definition problem. Our results on a real-scale network show that the proposed approach allows the solution to be obtained in less time but with the same accuracy as in traditional MSA approaches.

Keywords: Ant Colony Optimisation, signal settings, stochastic assignment.

1. INTRODUCTION

Generally, travel time of private system users can be split into three parts:

- running time, that is the time spent by vehicles travelling along roads;
- waiting time (delay) at intersections, that is the time spent crossing intersections;
- searching parking time, that is the time spent at destination looking for a parking space.

In urban contexts, the second term can make a substantial contribution to total travel time. Therefore, the effective optimisation of signalised intersections can significantly improve performance of private road systems.

The problem of optimising signal settings, generally indicated as the *Signal Setting Design Problem* (*SSDP*), can be considered as a particular case of the more general *Network Design Problem* (*NDP*) where the design variables are only the signal setting parameters (number of phases, cycle length, effective green times, etc.) while all other supply variables (such as widths or lane numbers) are fixed and invariable. In this kind of problem, link flows are descriptive variables, i.e. the analyst cannot directly modify them, but can influence them by changing design variables.

In the literature (see for instance Marcotte, 1993; Cantarella et al., 1991; Cantarella and Sforza, 1995; Cascetta et al., 2006; Cascetta, 2009) two kinds of approaches can be identified for solving the SSDP: the global approach and the local approach. In the first case, the problem, known as *Global Optimisation of Signal Settings* (*GOSS*), consists of calculating signal parameters by minimising an objective function, that is:

$$\hat{\boldsymbol{g}} = \arg\min_{\boldsymbol{g}} Z(\boldsymbol{g}, \boldsymbol{f}^*) \qquad (1)$$

subject to:

$$\boldsymbol{f}^* = \boldsymbol{\Lambda}(\boldsymbol{c}(\boldsymbol{g}, \boldsymbol{f}^*)) \qquad (2)$$

$$\boldsymbol{g} \in \boldsymbol{S}_g \qquad (3)$$

$$\boldsymbol{f}^* \in \boldsymbol{S}_f \qquad (4)$$

where:

$\boldsymbol{g}$ is the vector of signal setting parameters to be optimised;
$\hat{\boldsymbol{g}}$ is the optimal value of $\boldsymbol{g}$;
Z is the objective function, for instance the total travel time, to be minimised;
$\boldsymbol{f}^*$ is the equilibrium vector of link flows to be calculated by means of eqn (2);
$\boldsymbol{\Lambda}$ is the equilibrium assignment function;
$\boldsymbol{c}$ is the link cost vector depending on signal setting parameters (vector $\boldsymbol{g}$) and equilibrium flows (vector $\boldsymbol{f}^*$);
$\boldsymbol{S}_g$ is the feasibility set of vector $\boldsymbol{g}$;
$\boldsymbol{S}_f$ is the feasibility set of vector $\boldsymbol{f}^*$.

The first constraint (eqn. 2) represents the equilibrium assignment constraint that provides user flows ($\boldsymbol{f}^*$) as a function of link costs (vector $\boldsymbol{c}$) which are depend on design variables (vector $\boldsymbol{g}$) and equilibrium flows (vector $\boldsymbol{f}^*$). This constraint is formulated as a *fixed-point*

problem because user choices (and related flows) are generally influenced by network performance (i.e. users tend to choose the least cost alternatives) and transportation system performance is affected by user flows (i.e. an increase in on-road vehicles generally produces an increase in travel times).
According to constraint (3), signal setting parameters have to satisfy some conditions. For example, minimum green times, the sum of all effective green times, have to be equal to the effective cycle. Finally, constraint (4) expresses the fact that flows have to belong to feasibility sets that impose flow consistency (for instance, the sum of all incoming flows in a node has to be equal to the sum of all outgoing flows if the node is not a centroid).
A static approach to the GOSS problem is adopted by Sheffi and Powell (1983), Yang and Yagar (1995), Wong and Yang (1997), Chiou (1999), and Cascetta et al. (2006). The GOSS in the case of coordinated intersections is studied by Pillai et al. (1998) and Wey (2000). Moreover, a dynamic approach for the GOSS problem is proposed by Abdelfatah and Mahmassani (1999), and Abu-Lebdeh and Benekohal (2003).
The second approach of the SSDP, known as *Local Optimisation of Signal Settings* (*LOSS*), is based on assuming that signal settings at each intersection depend only on entering flows of the intersection according to a pre-specified control policy, such as the equisaturation method (Webster, 1958), the local delay minimization and Smith's P0 policy (Smith, 1980, 1981), or other isolated intersection optimisation methods (such as those of Chang and Lin, 2000; Wong and Wong, 2003). Hence, the LOSS problem can be formulated as:

$$\hat{\boldsymbol{g}} = \boldsymbol{\Omega}(\boldsymbol{f}^*) \tag{5}$$

subject to:

$$\boldsymbol{f}^* = \boldsymbol{\Lambda}(\boldsymbol{c}(\boldsymbol{g}, \boldsymbol{f}^*)) \tag{6}$$
$$\boldsymbol{g} \in \boldsymbol{S}_g \tag{7}$$
$$\boldsymbol{f}^* \in \boldsymbol{S}_f \tag{8}$$

where $\boldsymbol{\Omega}$ is the control policy which provides signal setting parameters as a function of equilibrium flows. In this case constraints (6), (7) and (8) assume the same meaning as constraints (2), (3) and (4).
Since control policy (5) has to provide a unique value of vector $\boldsymbol{g}$ for each value of equilibrium flow vector $\boldsymbol{f}^*$, the LOSS problem can be reformulated as a *fixed-point problem* indicated also as an *asymmetric equilibrium problem* (Cascetta et al., 2006), that is:

$$\boldsymbol{f}^* = \boldsymbol{\Lambda}(\boldsymbol{c}(\boldsymbol{\Omega}(\boldsymbol{f}^*), \boldsymbol{f}^*)) \tag{9}$$
$$\boldsymbol{f}^* \in \boldsymbol{S}_f \tag{10}$$

The asymmetric equilibrium problem has been extensively studied elsewhere: the static approach is analysed by Allsop (1974), Dafermos (1980, 1982), Florian and Spiess (1982), Smith and Van Vuren (1993), Lee and Hazelton (1996) and Cascetta et al. (2006), and the dynamic approach by Han (1996), Hu and Mahmassani (1997) and Lo et al. (2001). Finally, real-time applications of actuated signals are studied by Rakha (1993), Wolshon and Taylor (1999) and Mirchandani and Head (2001).

The LOSS approach vis-à-vis the GOSS has the advantage of solving signal settings optimisation with a significantly lower computing time, leading to a solution that is not significantly worse than that obtained by solving the GOSS problem (see Cascetta et al., 2006). This issue is of great importance for real-scale problems.

The aim of this paper is to provide a solution algorithm based on the *Ant Colony Optimisation* (*ACO*) paradigm for solving the SSDP in the case of the LOSS approach. Indeed, the solution of the asymmetric assignment problem requires more computing time than a classic assignment problem, since the signal settings have to be updated at each iteration and there is a double circular dependence (local optimal signal settings depend on flows that, in turn, depend on costs, that depend on both flows and signal settings). Therefore, we steered our research into ACO-based algorithms which, based on the food source search of ant colonies, have in many cases shown their efficiency in terms of calculation times (an extended overview of ACO-based algorithms can be found in Dorigo and Stützle, 2004).

Real world applications of the LOSS approach are mainly related to two cases: (a) to estimate the total travel time on the network produced at equilibrium by a system of actuated traffic signals; (b) to adopt it within urban network design models and algorithms for the joint calculation of equilibrium traffic flows and local optimal signal settings.

While reducing computing times in case (a) is not very important, since the solution has to be found only once, it is of great importance in case (b) where the solution has to be found many times inside the algorithm for solving the urban network design problem. An example of the urban network design model and algorithm that uses the LOSS approach for estimating traffic flows and signal settings can be found in Gallo et al. (2010), where a classic MSA-FA was adopted for solving the LOSS problem, leading to high computing times on real-scale networks. Indeed, the LOSS problem had be solved 52,735 times, resulting in over 113 hours spent on processing.

An exhaustive review of applications to transportation systems can be found in Teodorovic (2008). However, the main applications were proposed by Poorzahedy and Abulghasemi (2005) in the case of network design, by de Oliveira & Bazzan (2006) for traffic control, and D'Acierno et al. (2006) and Mussone et al. (2007) for traffic assignment.

The application of ACO theory in the case of (symmetric) assignment problems can be considered as an 'algorithmic trick' to speed up the calculation of the equilibrium solution (D'Acierno et al., 2006; Mussone et al., 2007). Hence, in this paper we modify the ACO algorithm proposed by D'Acierno et al. (2006) for solving the LOSS problem on real-scale networks, where reducing computing times is an important task.

This paper is organised as follows: Section 2 describes the general framework of ACO algorithms in the case of the traffic assignment problem; in Section 3, an ACO algorithm for solving the LOSS problem is proposed; in Section 4, a comparison among traditional algorithms and the proposed ACO algorithm is provided in the case of a real-scale network; Section 5 summarises conclusions and further research prospects; finally in Appendix A theoretical properties of the fixed-point problem and solution algorithms are described in detail.

2. THE ACO APPROACH IN THE TRAFFIC ASSIGNMENT PROBLEM

As shown in the previous section, a LOSS problem can be formulated by means of a fixed-point problem. Since D'Acierno et al. (2006) have shown that the use of artificial ants (i.e. the Ant Colony Optimisation approach) allows solution of a fixed-point problem to be obtained in less time but with the same accuracy compared with traditional algorithms, we propose to adopt the ACO approach for solving the LOSS problem.

In order to describe the main features of the ACO approach analytically, a short introduction to the behaviour of real and artificial ants is worth providing. In particular, in the case of real ants the search for food sources is based on the following simple rules:

- each ant provides a pheromone trail along its path;
- each ant follows a path if there is a pheromone trail;
- if there is no pheromone trail, ants choose their paths randomly;
- there is evaporation of pheromone trails, which causes short paths to have a more intense trail pheromone;
- if there is a diversion point (i.e. where different paths start), ants follow the path with the most intense pheromone trail.

Hence, if a path is blocked by an obstacle, ants initially choose their path randomly. Hence, due to evaporation, they choose to follow the path with the most intense pheromone trail, which is the shortest path.

Similarly, artificial ants can be described by these simple rules:

- the probability of choosing a path, indicated as *transition probability*, which depends on the intensity of the pheromone trail and, sometimes, a visibility term which expresses a sort of distance that could affect probability choices, that is:

$$p^t(l / i) = \frac{\left(\tau_l^t\right)^\alpha \cdot \left(\eta_l^t\right)^\beta}{\sum_{l' \in FS(i)} \left(\tau_{l'}^t\right)^\alpha \cdot \left(\eta_{l'}^t\right)^\beta} \tag{11}$$

  where:
  $p^t(l / i)$ is the probability of choosing link l, with l=(i, j), at diversion node i;
  τ_l^t $\left[\tau_{l'}^t\right]$ is the intensity of the pheromone trail on link l [l'] at iteration t;
  η_l^t $\left[\eta_{l'}^t\right]$ is the visibility term on link l [l'] at iteration t;
  $FS(i)$ is the set of links belonging to the forward star of node i;
  α and β are model parameters;

- the *pheromone increase* is the rule that expresses the quantity produced by each ant at any iteration, that is:

$$\Delta\tau_l^t = \lambda_l\left(\mathbf{p}^t, \mathbf{X}\right) \qquad (12)$$

where:

$\Delta\tau_l^t$ is the pheromone trail produced by each ant on link l at iteration t;

$\mathbf{p^t}$ is the transition probability matrix at iteration t whose generic element is $p^t(l / i)$;

$\mathbf{X}$ is the matrix of model parameters;

λ_l is the function that expresses the increase in the pheromone trail on link l depending on $\mathbf{p^t}$ and $\mathbf{X}$;

– the *pheromone trail update* is the rule governing the evaporation of pheromone trail and how the trail is increased by the contribution of each ant, that is:

$$\tau_l^t = (1-\rho)\cdot\tau_l^{(t-1)} + \rho\cdot\Delta\tau_l^t \qquad (13)$$

$$\left[\tau_l^t = (1-\rho)\cdot\tau_l^{(t-1)} + \Delta\tau_l^t\right] \qquad (14)$$

where ρ is the evaporation term.

Generally, a *traffic assignment problem* can be formulated as a fixed-point problem which can be obtained by combining the supply model with the demand model. The supply model describes transportation systems performance depending on user flows, that is:

$$\mathbf{C} = \mathbf{A}^{\mathrm{T}}\mathbf{c}(\mathbf{g},\mathbf{f}) + \mathbf{C}^{NA} \qquad (15)$$

where:

C is the vector of path generalised costs whose generic element C_k is the path cost related to path k;

A is the link-path incidence matrix whose generic element $a_{l,k}$ is equal to 1 if link l belongs to path k, 0 otherwise;

c is the vector of link costs whose generic element c_l is the cost of link l;

g is the matrix of signal setting parameters whose generic element $g_{l,h}$ is the h-th parameter (such as effective green or cycle length) related to link l;

f is the vector of link flows whose generic element f_l is the flow on link l;

$\mathbf{C^{NA}}$ is the vector of non-additive path costs whose generic element C_k^{NA} is the non-additive path cost of path k (such as: road tolls at motorway entrance/exit points or transit fares).

Likewise, the demand model imitates user choices influenced by transportation system performance, that is:

$$\mathbf{f} = \mathbf{A}\mathbf{P}(-\mathbf{C})\mathbf{d} \qquad (16)$$

where:

P is the matrix of path choice probabilities whose generic element $p_{k,od}$ expresses the probability of users travelling between origin-destination pair od choosing path k;

d is the demand vector whose generic element d_{od} expresses the average number of users travelling between origin-destination pair *od* in a time unit.

Therefore, the traffic assignment model can be obtained by combining eqn (15) with (16), that is:

$$\boldsymbol{f} = \boldsymbol{A}\boldsymbol{P}\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}) + \boldsymbol{C}^{NA}\right)\boldsymbol{d} \tag{17}$$

It is worth noting that eqn (17) represents the assignment constraint described by eqn (2) or equivalently eqn (6).
The existence and the uniqueness of the (equilibrium) solution of problem (17) may be stated, as shown by Cantarella (1997) and Cascetta (2009), if:

- choice probability functions, $\boldsymbol{P}(-\boldsymbol{C})$, are continuous;
- link cost functions, $\boldsymbol{c}(\boldsymbol{g}, \boldsymbol{f})$, are continuous;
- each origin-destination pair is connected (i.e. $I_{od} \neq \Phi \quad \forall\ od$);
- path choice models are expressed by strictly decreasing functions with respect to path generalised costs, that is:

$$\left[\boldsymbol{P}(-\boldsymbol{C}') - \boldsymbol{P}(-\boldsymbol{C}'')\right]^{\mathrm{T}}(\boldsymbol{C}' - \boldsymbol{C}'') < 0 \quad \forall \boldsymbol{C}' \neq \boldsymbol{C}'' \tag{18}$$

- cost functions are expressed by monotone non-decreasing functions with respect to link flows, that is:

$$\left[\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}') - \boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}'')\right]^{\mathrm{T}}(\boldsymbol{f}' - \boldsymbol{f}'') \geq 0 \quad \forall \boldsymbol{f}' \neq \boldsymbol{f}'' \tag{19}$$

Cantarella (1997) proposed to solve problem (17) by means of a solution algorithm known as the *Method of Successive Averages* (*MSA*). In particular, two algorithms, a *Flow Averaging* (*MSA-FA*) and a *Cost Averaging* (*MSA-CA*) algorithm, were proposed based respectively on the following sequences:

$$\boldsymbol{f}^{t+1} = \boldsymbol{f}^{t} + (1/t)\cdot\left(\boldsymbol{f}\left(\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}^{t})\right) - \boldsymbol{f}^{t}\right) \quad \text{with} \quad \boldsymbol{f}^{t} \in \boldsymbol{S_f} \tag{20}$$

$$\boldsymbol{c}^{t+1} = \boldsymbol{c}^{t} + (1/t)\cdot\left(\boldsymbol{c}\left(\boldsymbol{g},\boldsymbol{f}(\boldsymbol{c}^{t})\right) - \boldsymbol{c}^{t}\right) \quad \text{with} \quad \boldsymbol{c}^{1} = \boldsymbol{c}(\boldsymbol{f}^{1});\ \boldsymbol{f}^{t} \in \boldsymbol{S_f} \tag{21}$$

where $\boldsymbol{f}(.)$ is the network loading function described by eqn (16).
MSA algorithms stop when a stopping criterion is satisfied. In particular, in the case of MSA-FA and MSA-CA the stopping criterion is always:

$$\max_{l}\left|\frac{f_l\left(\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}^{t})\right) - f_l^{t}}{f_l^{t}}\right| < \varepsilon_{MSA} \tag{22}$$

where:
f_l^t is the link flow associated to link *l* at iteration *t*;

$f_l(.)$ is the link flow function which provides the flow of link l by means of eqn (16);
ε_{MSA} is the threshold used in the stopping criterion of MSA algorithms.

Both MSA algorithms can be used for solving the fixed point problem (17), providing the same results if the existence and uniqueness of the solution can be stated. Therefore, use of one algorithm rather than the other can be addressed only out of computational or implementing reasons.
Eqn (22) should represent the threshold that we accept as the maximum difference between the theoretical solution of problem (17) and the numerical result obtained by implementing a solution algorithm. Since we can solve problem (17) only numerically, we are not able to evaluate a priori the theoretical value of (17). Hence, we may adopt as a threshold the maximum difference between two successive iterations of the same algorithm. Indeed, at each iteration these algorithms generally provide asymptotically a solution ever closer to the theoretical value. Moreover, if we are able to implement two different algorithms, whose difference in solutions, evaluated by means of

$$max\left(\max_l \left| \frac{f_l^{Alg1} - f_l^{Alg2}}{f_l^{Alg1}} \right| \; ; \; \max_l \left| \frac{f_l^{Alg1} - f_l^{Alg2}}{f_l^{Alg2}} \right| \right) \tag{23}$$

where f_l^{Alg1} and f_l^{Alg2} are solution flows on link l of Algorithm 1 and Algorithm 2, provides a value lower than $(2 \cdot \varepsilon_{MSA})$, we may state that these algorithms converge to the same results.
However, it is necessary to fix a threshold value (value ε_{MSA} in eqn 22) which represents a fair trade-off between the accuracy of the solution and the calculation times involved. Moreover, it is worth noting that, since algorithm solutions represent vehicle flows on road network, in real cases it is not necessary to have a number of significant digits higher than a predetermined value related to the analysed problem. Hence, the evaluation of term ε_{MSA} is related to the input data accuracy.
By means of an extension of Blum's theorem (Blum, 1954), Cantarella (1997) stated the convergence of MSA algorithms described by eqns (20) and (21). Indeed, these algorithms satisfy convergence conditions as shown in Appendix A.
On analysing eqns (20) and (21), we may note that an assignment algorithm can be formulated as a sequence of network loadings. In particular, Dial's algorithm (Dial, 1971) is a stochastic network loading algorithm based on the following rules:

- let $Z_{d,i}$ be the minimum cost between node i and destination node d;
- a link $l = (i, j)$ is considered for a feasible path, indicated as a *Dial-efficient path*, only if:

$$Z_{d,i} > Z_{d,j} \tag{24}$$

- it is possible to associate a numerical quantity to each node and each link, indicated respectively as node weight and link weight, calculated as:

$$W_{d,d} = 1 \tag{25}$$

$$w_{d,(i,j)} = \begin{cases} exp\left(-c_{(i,j)}/\theta\right) \cdot W_{d,j} & \text{if } Z_{d,i} > Z_{d,j} \\ 0 & \text{if } Z_{d,i} \le Z_{d,j} \end{cases} \tag{26}$$

$$W_{d,i} = \sum_{(i,h) \in FS(i)} w_{d,(i,h)} \tag{27}$$

where:

$W_{d,i}$ is the node weight associated to node i in the case of destination node d;
$w_{d,(i,j)}$ is the link weight associated to link (i, j) in the case of destination node d;
$c_{(i,j)}$ is the link cost associated to link (i, j);
$FS(i)$ is the set of links belonging to the forward star of node i;

- the probability of choosing link $l = (i, j)$ at diversion node i is equal to:

$$p(l \,/\, i) = w_{d,(i,j)} / W_{d,i} \tag{28}$$

- the link flow f_l of generic link l can be calculated as:

$$f_{l=(i,j)} = \sum_d e_{d,(i,j)} = \sum_d E_{d,i} \cdot w_{d,(i,j)} / W_{d,i} \tag{29}$$

with:

$$E_{d,i} = \sum_{(h,j) \in BS(j)} e_{d,(h,j)} \tag{30}$$

$$E_{d,o} = d_{od} \tag{31}$$

where:

$BS(j)$ is the set of links belonging to the backward star of node j;
d_{od} is the average number of users travelling between origin-destination pair od in a time unit.

D'Acierno et al. (2006) proposed an ACO algorithm for solving the (symmetric) traffic assignment problem based on the following assumption:

- the initial intensity of the pheromone trail on each link l, associated to ant colony od, indicated as $\tau^0_{od,l}$, is a function of path costs, that is:

$$\tau^0_{od,l} = \sum_{k:l \in k} T^0_{od,k} \tag{32}$$

with:

$$T^0_{od,k} = \begin{cases} exp\left(-C_k^{\,0}/\theta\right) & \text{if } k \in I_{od} \\ 0 & \text{if } k \notin I_{od} \end{cases} \tag{33}$$

where:

$T^0_{od,k}$ is the initial intensity of the pheromone trail on path k;

$C_k^{\,0}$ is the initial cost of path k;

θ is the parameter of the path choice model;

I_{od} is the set of all available (or considered) paths that join origin node o with destination node d;

– the increase in the pheromone trail, indicated as $\Delta\tau^t_{od,l}$, can be expressed by a function of path costs, that is:

$$\Delta\tau^t_{od,l} = \sum_{k:l\in k} \Delta T^t_{od,k} \tag{34}$$

with:

$$\Delta T^t_{od,k} = \begin{cases} exp\left(-C_k^{\,t}/\theta\right) & \text{if } k \in I_{od} \\ 0 & \text{if } k \notin I_{od} \end{cases} \tag{35}$$

where:

$\Delta T^t_{od,k}$ is the increase in the pheromone trail at iteration t;

$C_k^{\,t}$ is the path k cost at iteration t;

– updating of the pheromone trail can be expressed as:

$$\tau^{t+1}_{od,l} = (1-\rho)\cdot\tau^t_{od,l} + \rho\cdot\Delta\tau^{t+1}_{od,l} \tag{36}$$

where evaporation coefficient ρ is variable and equal to $1/t$ (according to the approach proposed by Li and Gong, 2003).

In particular, D'Acierno et al. (2006) stated the theoretical equivalence of the proposed ACO algorithm to the application of an MSA algorithm where the successive averages were applied to the weight of Dial's algorithm (Dial, 1971), that is:

$$w^{t+1}_{d,(i,j)} = w^t_{d,(i,j)} + (1/t)\cdot\left(\Delta w^{t+1}_{d,(i,j)} - w^t_{d,(i,j)}\right) \tag{37}$$

$$W^{t+1}_{d,i} = W^t_{d,i} + (1/t)\cdot\left(\Delta W^{t+1}_{d,i} - W^t_{d,i}\right) \tag{38}$$

by fixing $\tau^{t+1}_{od,l} = w^{t+1}_{od,l}$ and $\Delta\tau^{t+1}_{od,l} = \Delta w^{t+1}_{od,l}$.

In this case, the MSA stopping criterion was:

$$\max_l \left| \frac{f_l(\boldsymbol{w}(\boldsymbol{c}(\boldsymbol{g}, \boldsymbol{f}^t))) - f_l^t}{f_l^t} \right| = \max_l \left| \frac{f_l(\boldsymbol{\tau}(\boldsymbol{c}(\boldsymbol{g}, \boldsymbol{f}^t))) - f_l^t}{f_l^t} \right| < \varepsilon_{MSA} \quad (39)$$

where:

$\boldsymbol{w}$ is the vector of Dial weights, whose generic element is $w_{d,(i,j)}$;

$\boldsymbol{\tau}$ is the vector of intensity of the pheromone trail, whose generic element is τ_l.

Finally, D'Acierno et al. (2006) stated the convergence of the proposed algorithm by means of the extension of Blum's theorem (Blum, 1954) provided by Cantarella (1997). Details of the convergence proof are reported in Appendix A.

3. THE PROPOSED ASSIGNMENT ALGORITHM

In order to develop an ACO-based algorithm for solving the LOSS problem, we modified artificial ants proposed by D'Acierno et al. (2006) by adding the following behaviour: at intersections each ant flow provides a pressure value which is directly proportional to the number of ants which travel in a time unit (flow) and inversely proportional to the 'net' width of their road (i.e. the road width reduced by the space occupied by parked vehicles). Therefore, in a simple case where there is an intersection with two entering one-way roads (as shown in Fig. 1), pressure values are:

$$\begin{cases} Press_1 = f_1 / L_1 \\ Press_2 = f_2 / L_2 \end{cases} \quad (40)$$

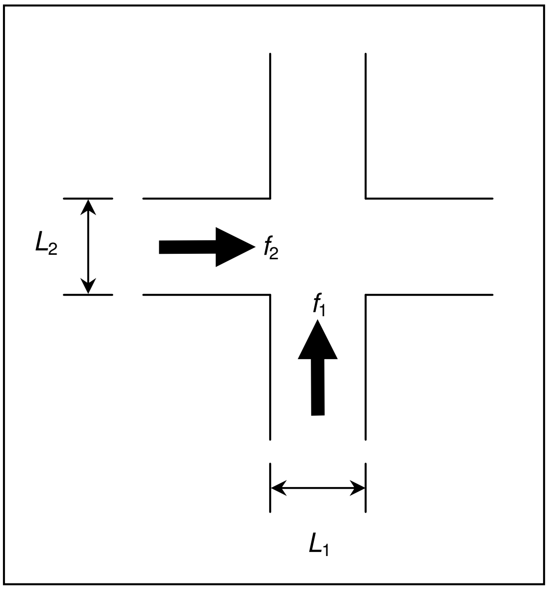

Figure 1 – Example of intersection with two entering one-way roads.

As in the case of two flows with the same priority level (for instance the case of people evacuation), the number of elements able to cross the intersection in a time unit is directly proportional to their pressure level. Therefore we may consider that each road has a shutter whose opening times are directly proportional to pressure levels, that is:

$$\begin{cases} \%time_1 = Press_1/(Press_1 + Press_2) \\ \%time_2 = Press_2/(Press_1 + Press_2) \end{cases} \quad (41)$$

In the case of an intersection with four bi-directional roads (as shown in Fig. 2), we may assume that influences between opposite roads can be neglected (the lower the flows of left turns, the more this assumption holds). In this case the pressure value which affects crossing phenomena can be calculated by means of the following relation:

$$\begin{cases} Press_1 = max(Press_N\,;Press_S) = max(f_N/L_N\,;f_S/L_S) \\ Press_2 = max(Press_E\,;Press_W) = max(f_E/L_E\,;f_W/L_W) \end{cases} \quad (42)$$

while the shutter operations can be described by eqn (41).

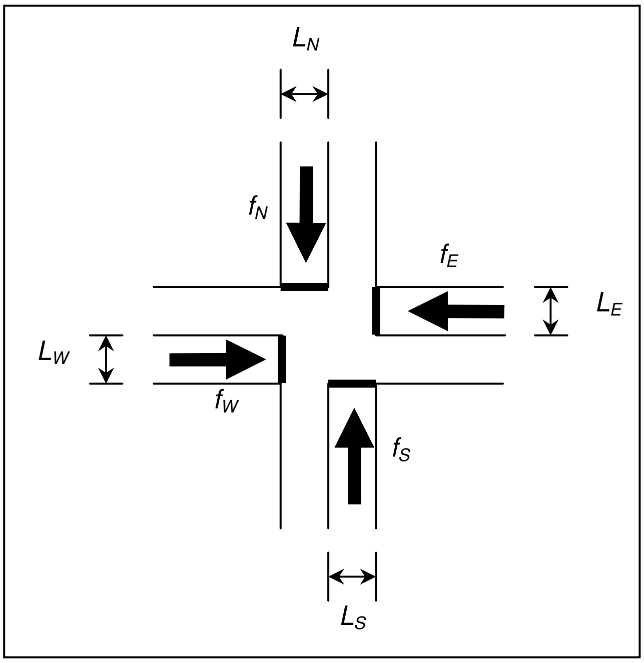

Figure 2 – Example of intersection with four bi-directional roads.

With reference to the behaviour of real road users, it may be stated that shutter operations are equivalent to a traffic light where signal setting parameters are calculated by means of the equisaturation method (Webster, 1958). Indeed, since saturation flows (i.e. the maximum flows which can cross an intersection with the assumption that traffic lights are always green) is directly proportional to the net width of the road, that is:

$$s_i \propto L_i \tag{43}$$

$\%time_i$ terms are directly proportional to saturation rates (where the saturation rate is the ratio between a flow and the related saturation), that is:

$$\begin{cases} \%time_1 = (f_1/s_1)/(f_1/s_1 + f_2/s_2) \\ \%time_2 = (f_2/s_2)/(f_1/s_1 + f_2/s_2) \end{cases} \tag{44}$$

According to the definition of shutter operations, term $\%time_i$ represents the effective green ratio, that is:

$$\begin{cases} \%time_1 = g_1/C_E \\ \%time_2 = g_2/C_E \end{cases} \tag{45}$$

where:
g_1 and g_2 are effective green lengths;
C_E is the difference between the cycle length and the total lost time per cycle.

By combining eqn (44) with eqn (45), we obtain:

$$\begin{cases} g_1/C_E = (f_1/s_1)/(f_1/s_1 + f_2/s_2) \\ g_2/C_E = (f_2/s_2)/(f_1/s_1 + f_2/s_2) \end{cases} \tag{46}$$

which is the analytical formulation of the equisaturation method for determining green distributions.
The modified behaviour of artificial ants allows simulation of the control policy $\boldsymbol{\Omega}$ for solving the LOSS problem.

3.1. Formulation of the algorithm for solving the LOSS problem

The MSA-FA algorithm for solving the LOSS problem proposed by Cascetta et al. (2006) can be described as follows:

Step 0: $k = 1$; $\boldsymbol{f}^1 = \boldsymbol{f}^0_{SNL}$

Step 1: $\boldsymbol{g}^k = \boldsymbol{\Omega}(\boldsymbol{f}^k)$

Step 2: $\boldsymbol{c}^k = \boldsymbol{c}(\boldsymbol{g}^k, \boldsymbol{f}^k)$

Step 3: $\boldsymbol{f}^k_{SNL} = \boldsymbol{AP}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}^k - \boldsymbol{C}^{NA})\boldsymbol{d}$

Step 4: $\boldsymbol{f}^{k+1} = ((k-1)\cdot \boldsymbol{f}^k + \boldsymbol{f}^{k+1}_{SNL})/k$

Step 5: If $\max_l \left\{ \left| f_l^{k+1} - f^k_{SNL,l} \right| / f_l^{k+1} \right\} < \varepsilon_{MSA}$ then go to Step 7

Step 6: $k = k + 1$; go to Step 1

Step 7: STOP

Likewise, it is possible to develop an MSA-CA algorithm for solving the LOSS problem. The algorithm features can be summarised as follows:

Step 0: $k = 1$; $\boldsymbol{g}^0 \in \boldsymbol{S}_g$; $\boldsymbol{f}^0 \in \boldsymbol{S}_f$; $\boldsymbol{c}^1 = \boldsymbol{c}(\boldsymbol{g}^0, \boldsymbol{f}^0)$; $\boldsymbol{f}^1_{SNL} = \boldsymbol{AP}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}^1 - \boldsymbol{C}^{NA})\boldsymbol{d}$

Step 1: $\boldsymbol{f}^k = \boldsymbol{f}^k_{SNL}$

Step 2: $\boldsymbol{g}^k = \boldsymbol{\Omega}(\boldsymbol{f}^k)$

Step 3: $\boldsymbol{y}^k = \boldsymbol{c}(\boldsymbol{g}^k, \boldsymbol{f}^k)$

Step 4: $\boldsymbol{c}^{k+1} = ((k-1)\cdot \boldsymbol{c}^k + \boldsymbol{y}^k)/k$

Step 5: $\boldsymbol{f}^{k+1}_{SNL} = \boldsymbol{AP}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}^{k+1} - \boldsymbol{C}^{NA})\boldsymbol{d}$

Step 6: If $\max_l \left\{ \left| f^{k+1}_{SNL,l} - f^k_l \right| / f^k_l \right\} < \varepsilon_{MSA}$ then go to Step 8

Step 7: $k = k + 1$; go to Step 1

Step 8: STOP

Finally, we may develop an ACO-based algorithm, indicated as MSA-ACO, for solving the LOSS problem that can be described as follows:

Step 0: $k = 1$; $\boldsymbol{g}^0 \in \boldsymbol{S}_g$; $\boldsymbol{f}^0 \in \boldsymbol{S}_f$; $\boldsymbol{c}^0 = \boldsymbol{c}(\boldsymbol{g}^0, \boldsymbol{f}^0)$; $\boldsymbol{\tau}^0 = \boldsymbol{\tau}(\boldsymbol{c}^0)$; $\boldsymbol{f}^1_{SNL} = \boldsymbol{f}(\boldsymbol{\tau}^0)$

Step 1: $\boldsymbol{f}^k = \boldsymbol{f}^k_{SNL}$

Step 2: $\boldsymbol{g}^k = \boldsymbol{\Omega}(\boldsymbol{f}^k)$

Step 3: $\boldsymbol{c}^k = \boldsymbol{c}(\boldsymbol{g}^k, \boldsymbol{f}^k)$

Step 4: $\boldsymbol{\Delta\tau}^k = \boldsymbol{\tau}(\boldsymbol{c}^k)$

Step 5: $\boldsymbol{\tau}^k = ((k-1)\cdot \boldsymbol{\tau}^{k-1} + \boldsymbol{\Delta\tau}^k)/k$

Step 6: $\boldsymbol{f}^{k+1}_{SNL} = \boldsymbol{f}(\boldsymbol{\tau}^k)$

Step 7: If $\max_l \left\{ \left| f^{k+1}_{SNL,l} - f^k_l \right| / f^k_l \right\} < \varepsilon_{MSA}$ then go to Step 9

Step 8: $k = k + 1$; go to Step 1

Step 9: STOP

where $\boldsymbol{\tau}^0 = \boldsymbol{\tau}(\boldsymbol{c}^0)$, $\boldsymbol{\Delta\tau}^k = \boldsymbol{\tau}(\boldsymbol{c}^k)$ and $\boldsymbol{f}^{k+1}_{SNL} = \boldsymbol{f}(\boldsymbol{\tau}^k)$ are respectively the synthetic formulations of eqns (32), (34) and (29), since $\boldsymbol{\tau}^k = \boldsymbol{w}^k$ and $\boldsymbol{\Delta\tau}^k = \boldsymbol{\Delta w}^k$.

As shown by Cascetta et al. (2006), the use of control policy $\boldsymbol{\Omega}$ does not allow us to state the uniqueness of equilibrium solution of problem (17) and hence related algorithm convergence.

4. FIRST RESULTS

The proposed ACO-based algorithm was applied on a real-scale network in order to compare its performance with that of traditional MSA algorithms. In this respect, Fig. 3 shows the network in Benevento (a town of about 62,000 inhabitants). Table I shows features of the analysed network, whereas Table II, as well as Figs. 4 and 5, indicates algorithm performance. In particular, 'FA solution error' in Table II represents the maximum percentage flow difference between the solution of the MSA-FA algorithm and that of the other algorithms, evaluated by means of:

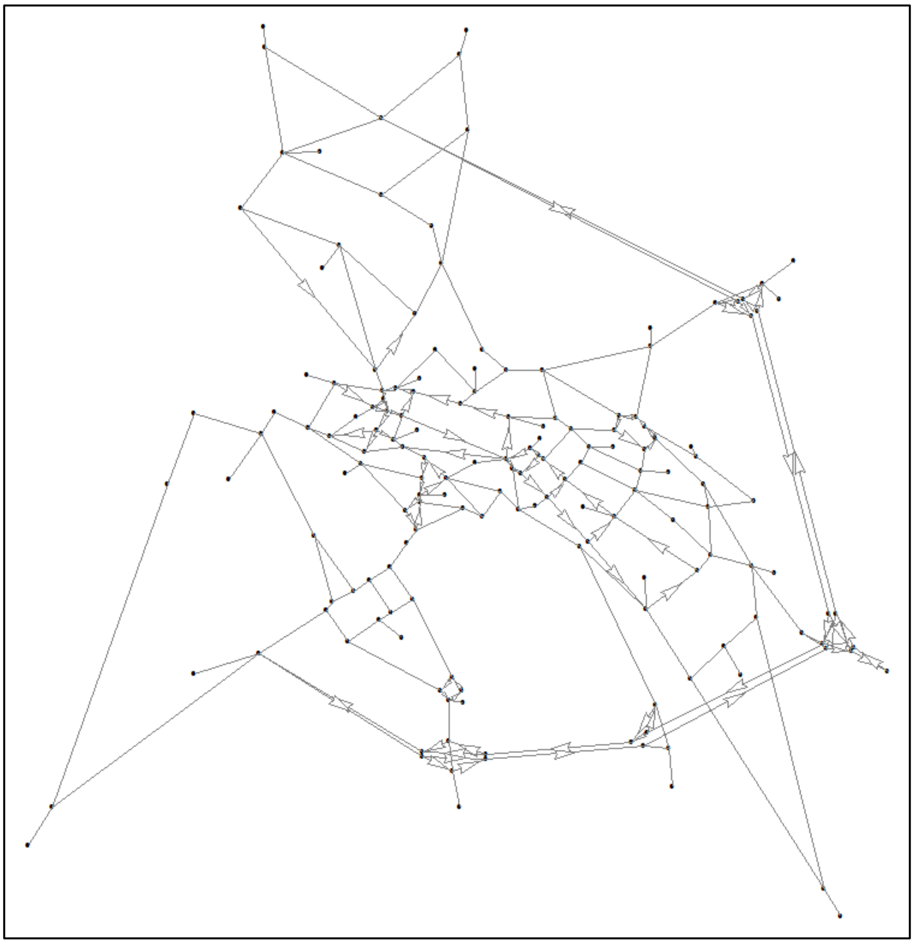

Figure 3 – Benevento network (Italy)

$$\max_l \left| \frac{f_l^{MSA-FA} - f_l^{MSA-CA}}{f_l^{MSA-FA}} \right| \quad (47)$$

$$\max_l \left| \frac{f_l^{MSA-FA} - f_l^{MSA-ACO}}{f_l^{MSA-FA}} \right| \quad (48)$$

where f_l^{MSA-FA}, f_l^{MSA-CA} and $f_l^{MSA-ACO}$ are flows on link l calculated respectively by means of MSA-FA, MSA-CA and MSA-ACO algorithms.
Likewise, 'CA solution error' in Table II can be calculated by means of:

$$\max_l \left| \frac{f_l^{MSA-CA} - f_l^{MSA-FA}}{f_l^{MSA-CA}} \right| \quad (49)$$

$$\max_l \left| \frac{f_l^{MSA-CA} - f_l^{MSA-ACO}}{f_l^{MSA-CA}} \right| \quad (50)$$

and 'ACO solution error' in Table II can be calculated by means of:

$$\max_l \left| \frac{f_l^{MSA-FA} - f_l^{MSA-ACO}}{f_l^{MSA-ACO}} \right| \quad (51)$$

$$\max_l \left| \frac{f_l^{MSA-CA} - f_l^{MSA-ACO}}{f_l^{MSA-ACO}} \right| \quad (52)$$

Table I – Network features

Number of links	Number of nodes	Number of centroid nodes	Number of OD pairs	Number of peak-hour trips
382	161	36	1,296	17,870

Table II – Algorithm performances

Algorithm name	Number of iterations	Convergence > 98.00%	Calculation Time [s]	FA solution error	CA solution error	ACO solution error
MSA-FA	321	73	15		6.48%	1.26%
MSA-CA	7	5	1	6.09%		5.25%
MSA-ACO	4	3	<1	1.25%	5.54%	

The proposed algorithm (termed MSA-ACO) is shown to provide the same solution as the algorithm proposed by Cascetta et al. (2006) (indicated in Table II as MSA-FA) in lower calculation times. Indeed, the difference in results, evaluated by applying eqn (23) to eqns (48) and (51), is 1.26% which is lower than ($2 \cdot \varepsilon_{MSA}$), since we fixed ε_{MSA} = 1.00%.

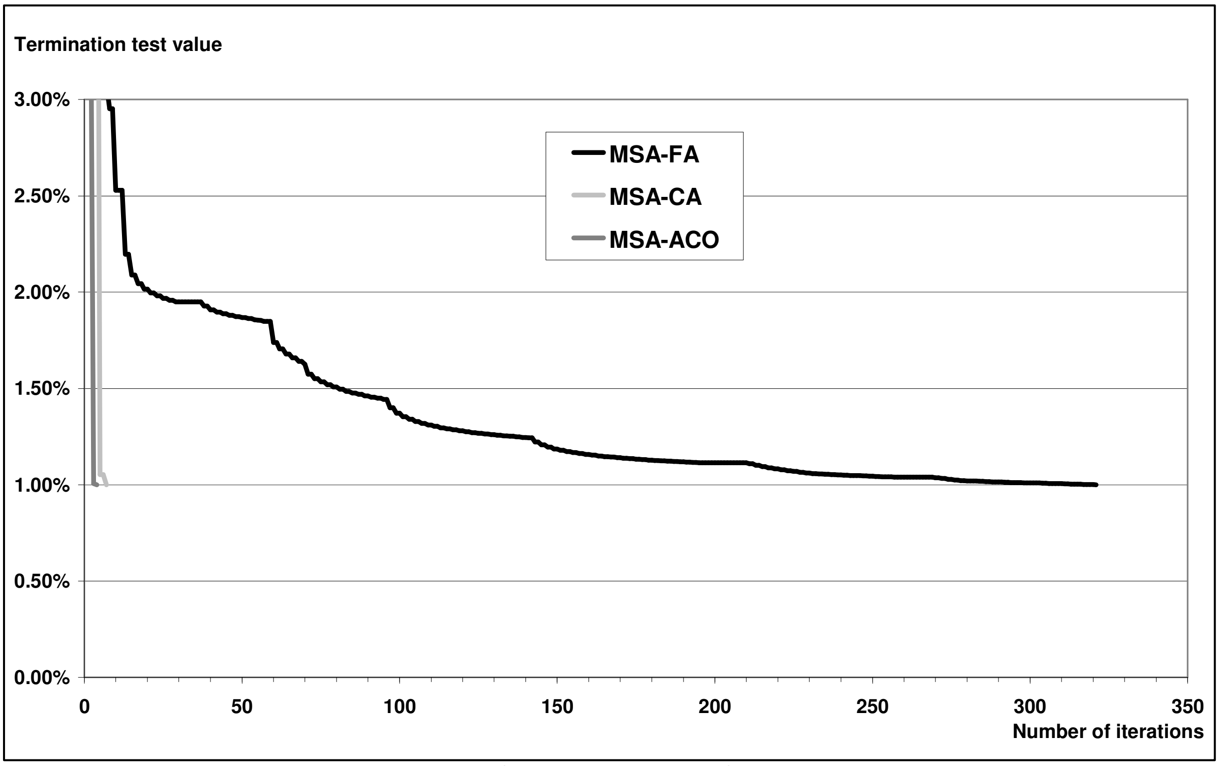

Figure 4 – Algorithm performances in terms of termination test values

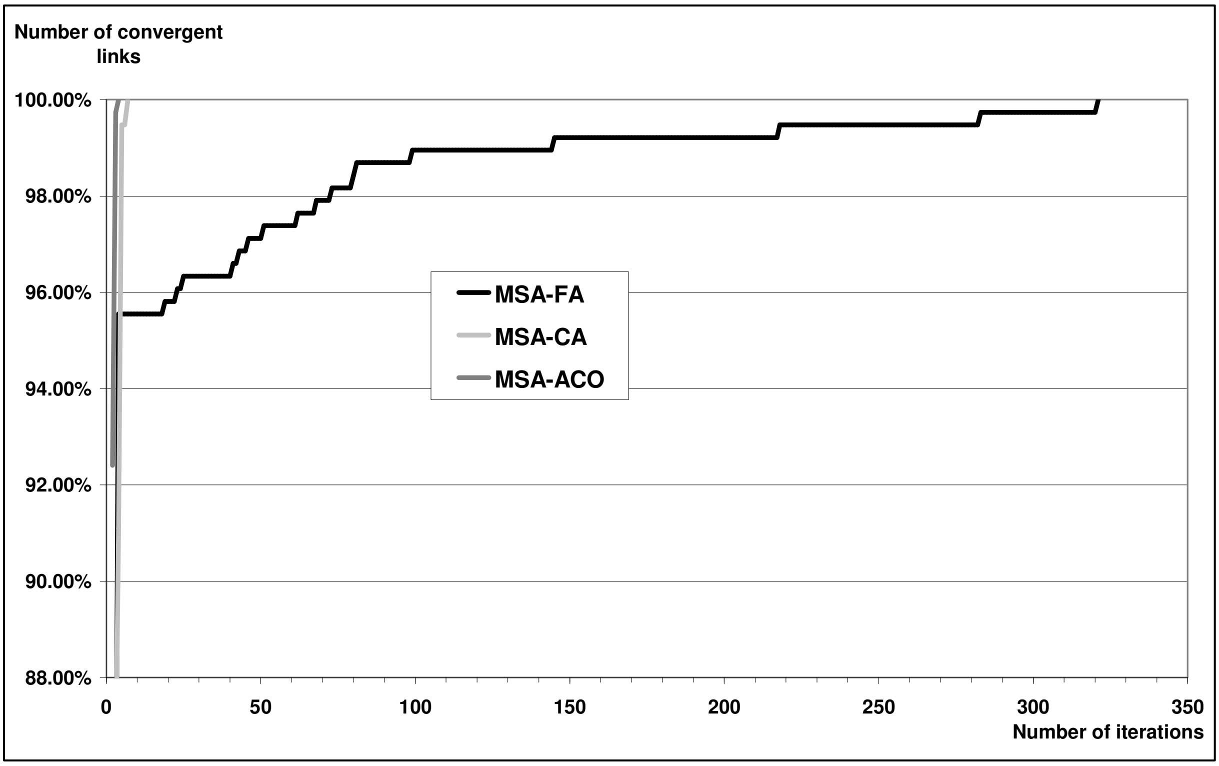

Figure 5 – Algorithm performances in terms of convergent links

Moreover, calculation times (expressed in terms of number of iterations) of the proposed algorithm are 98.75% lower if we want algorithm convergence to be achieved when all links satisfy the termination test. By contrast, if we accept only 98% of links satisfying the termination test, calculation times are reduced by 95.89%.
Interestingly, although MSA-CA allows a 97.82% reduction in calculation times with respect to the MSA-FA in the case of 100% convergent links and 93.15% in the case of 98% convergent links, it provides a higher solution error (6.48%) with respect to the proposed ACO algorithm. However, the ACO algorithm provides a 42.86% reduction in calculation time over the MSA-CA algorithm in the case of 100% convergent links and 40.00% in the case of 98% convergent links. In this case, the maximum difference in terms of solution error is 5.54%.
Finally, these results show that only MSA-FA and MSA-ACO provide the same results, since comparisons with MSA-CA provide respective differences of 6.48% and 5.54%, which are always higher than $(2 \cdot \varepsilon_{MSA})$. Therefore, the MSA-CA does not seem to be advisable for implementing real-scale networks.

5. CONCLUSIONS AND RESEARCH PROSPECTS

In this paper we proposed an ACO-based algorithm that can be used to solve the LOSS problem. In particular, in modifying the behaviour of artificial ants, we stated the perfect equivalence in terms of control policy between artificial ants (simulated with the proposed approach) and the equisaturation method (simulated with traditional algorithm of asymmetric traffic assignment). We also showed, in the case of a real-scale network, its numerical efficiency with respect to traditional MSA algorithms. In terms of future research, the proposed algorithm could be applied in various real-scale networks in order to verify whether its efficiency is maintained. Further, we propose to extend the model to the case of the GOSS problem and more complex path choice models (such as C-Logit and Probit). Finally, we advocate using the proposed algorithm as a simulation model for imitating the behaviour of transportation systems in more complex design problems or in real-time management in order to highlight the advantages of adopting an ACO approach.

APPENDIX A. THEORETICAL PROPERTIES OF THE FIXED-POINT PROBLEM AND SOLUTION ALGORITHMS

In this appendix, we propose the proof of existence and uniqueness of the equilibrium solution proposed by Cantarella (1997) and Cascetta (2009). Moreover, we describe in detail the extension of Blum's theorem (Blum, 1954) provided by Cantarella (1997) in order to state the convergence of solution algorithms.

A.1. Existence of equilibrium solution

Cantarella (1997) and Cascetta (2009) stated that the fixed point problem:

$$\boldsymbol{f} = \boldsymbol{AP}\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}) + \boldsymbol{C}^{NA}\right)\boldsymbol{d} \tag{A.1}$$

has at least one solution if:

- choice probability functions, $\boldsymbol{P}(-\boldsymbol{C})$, are continuous;
- link cost functions, $\boldsymbol{c}(\boldsymbol{g}, \boldsymbol{f})$, are continuous;
- each origin-destination pair is connected (i.e. $I_{od} \neq \Phi \quad \forall\ od$).

<u>Proof.</u>
The equilibrium solution $\boldsymbol{f}$ is a fixed point of the compound function $\boldsymbol{y} = \boldsymbol{AP}\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{x}) + \boldsymbol{C}^{NA}\right)\boldsymbol{d} = \boldsymbol{\psi}(\boldsymbol{x})$, which under the above assumption is a continuous function defined over the non-empty, compact and convex set $\boldsymbol{S_f}$ (since each OD pair is connected). Furthermore, the function $\boldsymbol{y} = \boldsymbol{\psi}(\boldsymbol{x})$ assumes values only in the definition set $\boldsymbol{S_f}$, thus satisfying all the assumptions of Brouwer's theorem on the existence of fixed points. Hypotheses of Brouwer's theorem are reported below for the reader's convenience.

A.1.1. Hypotheses of Brouwer's theorem

A fixed-point problem $\boldsymbol{y} = \boldsymbol{\psi}(\boldsymbol{y})$ has at least one solution, i.e. the function $\boldsymbol{\psi}(\boldsymbol{x})$ defined in the set $\boldsymbol{S} \subseteq \boldsymbol{E}^n$ with values in the set $\boldsymbol{T} = \boldsymbol{\psi}(\boldsymbol{S}) \subseteq \boldsymbol{E}^n$ has at least one fixed-point if:

- $\boldsymbol{T}$ is a subset of $\boldsymbol{S}$, $\boldsymbol{T} \subseteq \boldsymbol{S}$, i.e. $\boldsymbol{\psi}(\boldsymbol{x}) \subseteq \boldsymbol{S} \quad \forall\ \boldsymbol{x} \in \boldsymbol{S}$;
- $\boldsymbol{S}$ is a compact and a convex non-empty set;
- $\boldsymbol{\psi}(\boldsymbol{x})$ is a continuous function.

A.2. Uniqueness of equilibrium solution

Cantarella (1997) and Cascetta (2009) proved that the fixed-point problem (A.1) has at most one solution if:

- path choice models are expressed by strictly decreasing functions with respect to path generalised costs, that is:

$$\left[\boldsymbol{P}(-\boldsymbol{C}') - \boldsymbol{P}(-\boldsymbol{C}'')\right]^{\mathrm{T}}(\boldsymbol{C}' - \boldsymbol{C}'') < 0 \quad \forall \boldsymbol{C}' \neq \boldsymbol{C}'' \tag{A.2}$$

- cost functions are expressed by monotone non-decreasing functions with respect to link flows, that is:

$$\left[\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}') - \boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}'')\right]^{\mathrm{T}}(\boldsymbol{f}' - \boldsymbol{f}'') \geq 0 \quad \forall \boldsymbol{f}' \neq \boldsymbol{f}'' \tag{A.3}$$

<u>Proof.</u>
By multiplying eqn (A.2) by constant quantities, that is $\boldsymbol{A}$ and $\boldsymbol{d}$, we obtain:

$$\left[\boldsymbol{AP}(-\boldsymbol{C}')\boldsymbol{d} - \boldsymbol{AP}(-\boldsymbol{C}'')\boldsymbol{d}\right]^{\mathrm{T}}(\boldsymbol{C}'-\boldsymbol{C}'') < 0 \quad \forall \boldsymbol{C}' \neq \boldsymbol{C}'' \tag{A.4}$$

Moreover, by replacing eqn (15), we can rewrite the above equation as:

$$\left[\boldsymbol{P}\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}') - \boldsymbol{C}^{NA}\right) - \boldsymbol{P}\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}'') - \boldsymbol{C}^{NA}\right)\right]^{\mathrm{T}}\left[\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}') - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}'')\right] < 0 \quad \forall \boldsymbol{f}' \neq \boldsymbol{f}'' \ \text{ with } \ \boldsymbol{f} = \boldsymbol{AP}(-\boldsymbol{C}'')\boldsymbol{d} \tag{A.5}$$

Then, by multiplying the first term by a positive quantity, that is $\boldsymbol{d}$, we obtain:

$$\left[\boldsymbol{AP}\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}') - \boldsymbol{C}^{NA}\right)\boldsymbol{d} - \boldsymbol{AP}\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}'') - \boldsymbol{C}^{NA}\right)\boldsymbol{d}\right]^{\mathrm{T}}\left[\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}') - \boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}'')\right] < 0 \quad \forall \boldsymbol{f}' \neq \boldsymbol{f}'' \ \text{ with } \ \boldsymbol{f} = \boldsymbol{AP}(-\boldsymbol{C})\boldsymbol{d} \tag{A.6}$$

By means of eqn (16), we obtain:

$$(\boldsymbol{f}'-\boldsymbol{f}'')^{\mathrm{T}}\left[\boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}') - \boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}'')\right] < 0 \quad \forall \boldsymbol{f}' \neq \boldsymbol{f}'' \ \text{ with } \ \boldsymbol{f} = \boldsymbol{AP}(-\boldsymbol{C})\boldsymbol{d} \tag{A.7}$$

At this point, the proof is then completed by *reductio ad absurdum*. Indeed, if two different equilibrium link flows existed, $\boldsymbol{f}_1^* \neq \boldsymbol{f}_2^* \in \boldsymbol{S}_f$, assuming $\boldsymbol{c}_1^* = \boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}_1^*)$ and $\boldsymbol{c}_2^* = \boldsymbol{c}(\boldsymbol{g},\boldsymbol{f}_2^*)$, eqn (A.7) would yield:

$$\left(\boldsymbol{f}_1^* - \boldsymbol{f}_2^*\right)^{\mathrm{T}}\left(\boldsymbol{c}_1^* - \boldsymbol{c}_2^*\right) < 0 \tag{A.8}$$

while, from eqn (A.3), it follows that:

$$\left(\boldsymbol{f}_1^* - \boldsymbol{f}_2^*\right)^{\mathrm{T}}\left(\boldsymbol{c}_1^* - \boldsymbol{c}_2^*\right) \geq 0 \tag{A.9}$$

Thus, there is a contradiction between (A.2) and (A.3).

A.3. Convergence of solution algorithms

Cantarella (1997) showed, by adapting Blum's theorem (Blum, 1954) to the fixed-point problem (A.1), that a convergent solution algorithm for solving the fixed-point problem of a function $\boldsymbol{y} = \boldsymbol{\psi}(\boldsymbol{x})$ with:

- $\boldsymbol{x} \in \boldsymbol{S}_x$ and $\boldsymbol{y} \in \boldsymbol{S}_x$ where $\boldsymbol{S}_x$ is a non-empty, compact and convex set;
- a unique fixed-point $\boldsymbol{x}^* = \boldsymbol{\psi}(\boldsymbol{x}^*)$;
- a function $\varphi(\boldsymbol{x}) \geq 0 \quad \forall \boldsymbol{x} \in \boldsymbol{S}_x$ where $\varphi(\boldsymbol{x})$ is continuous with first $\nabla\varphi(\boldsymbol{x})$ and second $\nabla^2\varphi(\boldsymbol{x})$ derivative continuous;
- $\nabla^2\varphi(\boldsymbol{x})^T[\boldsymbol{\lambda}(\boldsymbol{x}) - \boldsymbol{x}] < 0 \quad \forall \boldsymbol{x} \in \boldsymbol{S}_x, \boldsymbol{x} \notin \boldsymbol{S}_{\tilde{x}}$ and $\nabla^2\varphi(\tilde{\boldsymbol{x}})^T[\boldsymbol{\lambda}(\tilde{\boldsymbol{x}}) - \tilde{\boldsymbol{x}}] = 0 \quad \forall \tilde{\boldsymbol{x}} \in \boldsymbol{S}_{\tilde{x}}$ where $\boldsymbol{S}_{\tilde{x}} \subseteq \boldsymbol{S}_x$ and $\boldsymbol{x}^* \in \boldsymbol{S}_{\tilde{x}}$;
- $\left|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}^*)\right| > 0 \quad \forall \boldsymbol{x} \in \boldsymbol{S}_x, \boldsymbol{x} \notin \boldsymbol{S}_{\tilde{x}}$ and $\left|\varphi(\tilde{\boldsymbol{x}}) - \varphi(\boldsymbol{x}^*)\right| = 0 \quad \forall \tilde{\boldsymbol{x}} \in \boldsymbol{S}_{\tilde{x}}$;

– $\boldsymbol{x}^T \nabla^2 \varphi(\boldsymbol{x}')\boldsymbol{x} = M < +\infty \quad \forall \boldsymbol{x}, \boldsymbol{x}' \in \boldsymbol{S}_x$;

can be formulated by the following recursive equation:

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t + \mu_t\left(\boldsymbol{\lambda}(\boldsymbol{x}^t) - \boldsymbol{x}^t\right) \qquad \text{with } \boldsymbol{x}^t \in \boldsymbol{S}_x \tag{A.10}$$

if the sequence $\{\mu_t\}_{t>0}$ satisfies the following conditions:

$$\sum_{t>0} \mu_t = +\infty, \quad \sum_{t>0} (\mu_t)^2 = M < +\infty \tag{A.11}$$

Moreover, if the sequence $\{\mu_t\}_{t>0}$ satisfies the condition:

$$\mu_t \in \left]0,1\right] \tag{A.12}$$

then the elements of the sequence described by (A.10) belong to set $\boldsymbol{S}_x$, which is convex.
A sequence $\{\mu_t\}_{t>0}$ which satisfies both (A.11) and (A.12) is given by $\{\mu_t = 1/t\}_{t>0}$ such that (A.10) becomes:

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t + \frac{1}{k}\left(\boldsymbol{\lambda}(\boldsymbol{x}^t) - \boldsymbol{x}^t\right) = \left(1 - \frac{1}{t}\right)\boldsymbol{x}^t + \frac{1}{t}\boldsymbol{\lambda}(\boldsymbol{x}^t) \tag{A.13}$$

With the above assumption we may develop a solution algorithm known as the Method of Successive Averages (MSA). In particular, Cantarella (1997) proposed a Flow Averaging algorithm (MSA-FA) based on the following sequence:

$$\boldsymbol{f}^{t+1} = \boldsymbol{f}^t + \frac{1}{t}\left(\boldsymbol{f}(\boldsymbol{c}(\boldsymbol{f}^t)) - \boldsymbol{f}^t\right) \in \boldsymbol{S}_f \quad \text{with } \boldsymbol{f}^1 \in \boldsymbol{S}_f \tag{A.14}$$

which corresponds to eqn (20), and a Cost Averaging algorithm (MSA-CA) based on:

$$\boldsymbol{c}^{t+1} = \boldsymbol{c}^t + \frac{1}{t}\left(\boldsymbol{c}(\boldsymbol{f}(\boldsymbol{c}^t)) - \boldsymbol{c}^t\right) \in \boldsymbol{S}_c \quad \text{with } \boldsymbol{c}^1 = \boldsymbol{c}(\boldsymbol{f}^1) \in \boldsymbol{S}_c \text{ and } \boldsymbol{f}^1 \in \boldsymbol{S}_f \tag{A.15}$$

which corresponds to eqn (21).
Moreover, in order to prove the convergence of algorithms, assuming that existence and uniqueness conditions hold, it is necessary to verify that link cost functions have a symmetric continuous Jacobian $Jac[\boldsymbol{c}(\boldsymbol{f})]$ over set $\boldsymbol{S}_f$, for MSA-FA and choice map functions, which are expressed by (16), are additive and continuous with continuous first derivative for algorithm MSA-CA. Importantly, these conditions are generally satisfied by almost all functions proposed in the literature.
Finally, D'Acierno et al. (2006) stated the convergence of the ACO algorithm described in Section 2, assuming that the existence and uniqueness condition holds, if $Jac[\boldsymbol{c}(\boldsymbol{\tau})]$ is symmetric and continuous. Indeed, this condition satisfies all hypotheses of Blum's theorem

(Blum, 1954) in the case of the fixed-point problem $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{c}(\boldsymbol{f}(\boldsymbol{\tau})))$ where $\boldsymbol{\tau}$ is a vector whose generic element is the pheromone trail $\tau_{od,l}$ (or equivalently $w_{od,l}$), of dimensions $((n_{Pairs} \cdot n_{Links}) \times 1)$. Moreover, sufficient conditions to verify the $Jac[\boldsymbol{c}(\boldsymbol{\tau})]$ hypothesis are that link cost functions have a symmetric and continuous Jacobian $Jac[\boldsymbol{c}(\boldsymbol{f})]$ over set $\boldsymbol{S_f}$, and choice map functions, which are expressed by (16), are additive and continuous with the continuous first derivative. Since also in this case these sufficient conditions are generally satisfied by almost all functions proposed in the literature, convergence of the proposed ACO-based algorithm may be postulated.

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