SENSITIVITY ANALYSIS BASED APPROACH FOR VULNERABILITY ANALYSIS OF LARGE-SCALE ROAD NETWORK

LUATHEP, Paramet. Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Kowloon, Hong Kong, China

SUMALEE Agachai. Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Kowloon, Hong Kong, China (ceasumal@polyu.edu.hk)

HO, H.W. Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Kowloon, Hong Kong, China

KURAUCHI, Fumitaka. Department of Civil Engineering, Gifu University, Gifu City, Japan

ABSTRACT

Vulnerability analysis is vital in strategic transport network planning to cope with the impacts from natural or malevolent events. Traditionally, an assessment of network vulnerability is a highly computational-intensive operation. This paper proposes a sensitivity analysis-based (SA) approach to improve the computational efficiency and allow for the large-scale application of network vulnerability analysis. Different vulnerability indicators could be used by the proposed method. For illustrative purpose, this paper adopts the relative accessibility index (AI), which follows the Hansen integral index, as the network vulnerability measure for evaluating the socio-economic impact from road (or link) capacity degradations or closures. Under network disruption, links with a large change of AI are more critical. Critical links are then identified by ranking the links in accordance to the change of AI. The proposed method only requires a single calculation of network equilibrium condition that extremely reliefs the computational burden and storage requirement of the traditional approach. The networks of Sioux Falls city and Bangkok metropolitan area are used to demonstrate the feasibility and efficiency of the proposed technique. Network managers and policy-makers can use the proposed scheme as a decision-supporting tool for identifying critical links. By improving the critical links or constructing new by-pass roads (or parallel routes) as a preventive action, the overall vulnerability of the network could be reduced.

Keywords: vulnerability analysis, sensitivity analysis, large-scale

1 INTRODUCTION

1.1 Background

The major roles of transport network are to provide accessibility, to connect urban centres, and to provide regional coverage and basic necessities for non-urban areas. Apart from these, the transport network should also be able to maintain the accessibility of the region under unexpected and expected impacts from natural or malevolent events.

Earlier studies have focused on evaluating transport network reliability which can be defined as a capability of the network to handle the variation of demand (Asakura and Kashiwadani, 1991), and/or supply (Chen *et al*., 1999). Various measures for assessing network reliability have been proposed: travel time reliability (Bell and Iida, 1997), flow decrement reliability (Du and Nicholson, 1997), capacity reliability (Chen *et al*., 2002), and connectivity reliability (Wakabayashi and Iida, 1992). Most of the reliability studies rely on the presumption that the information on the probability of degradation events is available. However, this information is difficult to obtain in practice particularly for a rare event, e.g. earthquake or tsunami. These rare events often result in a large-scale disruption with severe social and economic impacts. D'Este and Taylor (2003) suggested that network reliability based on link choice probabilities may not be appropriate to evaluate the impacts of such rare events.

Vulnerability analysis is a proactive approach for identifying weak spots in a network and evaluating the adverse consequences from network failure. The vulnerability analysis emphasizes on the consequences from network degradation or failure and avoids using the event probability information in the analysis. The vulnerability analysis is thus vital in strategic transport network planning for dealing with the impacts from natural or malevolent events. The result of vulnerability analysis can be used to recommend proactive remedial countermeasures, e.g. improving the performance of the vulnerable (or weak) links to be more robust, or adding new links to provide more alternative routes. From the literatures, two parallel streams of the research on transport network vulnerability have been conducted. The first one is the introduction of theories and definitions of network vulnerability. The second one is the development of algorithms and applications of vulnerability analysis.

Transport network vulnerability was initially introduced by Berdica (2002). From the literatures, two definitions are commonly. One is connectivity vulnerability, which is the ease of two adjacent nodes to be connected for activities and services (Bell and Iida, 1997). The other uses the accessibility index, which was originally proposed by Hansen (1959), to define accessibility vulnerability (D'Este *et al*., 2003). Subsequent studies have been conducted to precisely define and formulate the network vulnerability. Chen *et al*. (2007) used an increase in network travel time (or generalized travel cost) as the network-based accessibility measure for assessing the consequence of one or more link failures. They also adopted a combined travel demand model for determining the long-term equilibrium condition due to network disruption. Their results showed that the accessibility measure derived from the combined model can be used to determine the network impacts from both demand and supply changes. Their model also reflects the effects of travel time choice dimensions on the

network vulnerability. Recently, several alternative approaches and indicators have been proposed for vulnerability analysis. Lleras-Echeverri and Sanchez-Silva (2001) introduced a critical-scenario (CS)-based approach for the functional classification of links in the network. Critical link is defined as a link whose failure causes the largest increase in generalized network travel cost. Jenelius *et al*. (2006) introduced the link importance and site exposure indices for identifying critical links. Two groups of vulnerability measures, including equal opportunities perspectives and social efficiency perspective, were considered in their study.
Regarding the algorithms and applications, several works applied the vulnerability analysis to real world road networks (Berdica and Mattsson, 2007, Jenelius, 2009, Taylor et al., 2006, Taylor and D'Este, 2007, Taylor, 2008, D'Este *et al*., 2003). Taylor (2008) adopted the accessibility index for assessing the system-wide impact of traffic incidents in urban road network. From the literatures, computational efficiency has been recognized as the most challenging issue for conducting the vulnerability analysis with large-scale networks. The traditional vulnerability analysis involves completely removing the link(s), or partially decreasing the capacity of link(s) in the network, and solves for the network equilibrium condition with the degraded network. This approach is computationally burdensome and requires large memory storage.

A general idea of vulnerability analysis is to assess the implicit relationship between input data, such as link capacity degradation(s) or link closure(s), and the change of network performance, such as network travel time or generalized travel cost, or other network vulnerability measures. This concept is similar to the principle of sensitivity analysis (SA). The SA has a long history in both non-linear programming and transport network analysis. Various applications of SA technique were made in transport fields: trip matrix estimation (Yang *et al*., 1992), capacity design problem (Magnanti and Wong, 1984, Patriksson and Rockafellar, 2002), toll design problem (Connors *et al*., 2007, Sumalee *et al*., 2006a, Yang and Bell, 1997), signal control problem (Chow and Lo, 2007), network reliability assessment (Chen *et al*., 2002, Sumalee and Kurauchi, 2006, Sumalee *et al*., 2009, Sumalee *et al*., 2006b), and error estimation (Leurent, 1998, Nakayama *et al*., 2009). Despite these extensive applications, there are few studies which adopt the SA method for vulnerability analysis.

SA technique deals with the implicit relationship between the input data of a traffic assignment model and the equilibrium network flows based on that given data. However, the directional derivatives of the deterministic user equilibrium (UE) condition may not exist at certain points (Robinson, 2006). To avoid this problem, a stochastic user equilibrium (SUE) has been proposed as an alternative network flow model. Two types of SUE models, including logit and probit, have been extensively studied with the SA. The SA of logit-based SUE exists everywhere under mild condition (Davis, 1994, Ying and Miyagi, 2001). However, the logit model cannot represent the correlation of alternative routes in the network. On the other hand, the probit-based SUE, proposed by Daganzo and Sheffi (1977), takes into account of the joint distribution of alternative routes in the network. Clark and Watling (2002) clearly explained and derived the SA of probit SUE under fixed demand. Connors et al. (2007) extended the works of Clark and Watling (2002) to the case of variable demand with multiple user classes.

1.2 Aim of the paper

This paper focuses on the methodological development for assessing network vulnerability. We utilize the SA method to improve the computational efficiency of vulnerability analysis and allow for large-scale applications. We derive all gradients based on the probit SUE following Clark and Watling (2002) and Connors *et al*. (2007).

The proposed method can be applied with other indicators of network vulnerability. For illustrative purpose, we use the relative accessibility index (AI), following a normalised form of Hansen integral accessibility index (Davidson, 1977), as the network vulnerability measure to represent the socio-economic impact from partial link capacity degradations or complete link closures. The SA method will evaluate the implicit relationship of capacity degradations (or link closures) to the change of AI. Under degraded network condition, a link with a larger change of AI is more critical. Critical links are then identified by ranking the links according to the changes of AI. Note that the proposed method only requires a single computation of the network equilibrium condition. The SA method thus significantly reduces the computational burden of large-scale network analysis as compared to the traditional approach.

The remainder of the paper is outlined as follows. Section2 presents some basic notations and network representation. Section 3 proposes sensitivity analysis method for vulnerability analysis. In Section 4, two real world networks are used to demonstrate the feasibility and efficiency of the proposed method. The final section concludes the paper and discusses future research issues.

2 NOTATION AND NETWORK REPRESENTATION

2.1 Notation

The road network is represented by a directed graph. The following notations are used throughout the paper.

- N set of nodes
- R set of origin nodes, $R \subset N$
- S set of destination nodes, $D \subset N$
- r an origin node, $\forall r \in R$
- s a destination node, $\forall s \in S$
- W set of origin to destination (OD) movements
- rs an origin to destination (OD) pair, $\forall rs \in W$
- K set of paths
- k a path index, $\forall k \in K$
- A set of links
- a a link index, $\forall a \in A$
- $\delta_{a,k}^{rs}$ a link-path incident indicator, $\delta_{a,k}^{rs} = 1$ if path k connecting the OD pair rs uses link a, $\delta_{a,k}^{rs} = 0$ otherwise

Δ block-diagonal link-path incidence matrix with size $\left[A \times W\right]$,

$\Delta = \begin{bmatrix} \Delta^1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Delta^2 & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \Delta^{rs} \end{bmatrix}$ where Δ^{rs} is the $\left[A \times K^{rs}\right]$ element link-path incident matrix of OD pair rs and whose components are $\delta_{a,k}^{rs}$.

$\mathbf{q}$ column vector of travel demands with size $\left[W \times 1\right]$, $\mathbf{q} = \left[q^1, q^2, ..., q^{rs}\right]^T$, $\forall rs \in W$ where q^{rs} is the travel demand of OD pair rs.

$\mathbf{P}$ block-column vector of path choice probability with size $\left[W \times 1\right]$, $\mathbf{P} = \left[\mathbf{P}^1, \mathbf{P}^2, ..., \mathbf{P}^{rs}\right]^T$, $\forall rs \in W$ where $\mathbf{P}^{rs} = \left[P_1^{rs}, P_2^{rs}, \ldots, P_k^{rs}\right]^T$ is the $\left[K^{rs} \times 1\right]$ column vector of probabilities of the paths serving the OD pair rs.

$\mathbf{f}$ block-column vector of path flows with size $\left[W \times 1\right]$, $\mathbf{f} = \left[\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^{rs}\right]^T$, $\forall rs \in W$ where $\mathbf{f}^{rs} = \left[f_1^{rs}, f_2^{rs}, \ldots, f_k^{rs}\right]^T$ is the $\left[K^{rs} \times 1\right]$ column vector of flows of the paths serving the OD pair rs and each path flow is defined as $f_k^{rs} = P_k^{rs} q^{rs}$.

$\mathbf{x}$ column vector of link flows with size $\left[A \times 1\right]$, $\mathbf{x} = \left[x_1, x_2, ..., x_a\right]^T$, $\forall a \in A$ and the vector of link flow is defined as $\mathbf{x} = \sum_{rs \in W} \Delta^{rs} \cdot \left(q^{rs} \cdot \mathbf{P}^{rs}\right)$.

$\mathbf{y}$ column vector of original capacities with size $\left[A \times 1\right]$, $\mathbf{y} = \left[y_1, y_2, ..., y_a\right]^T$, $\forall a \in A$.

$\mathbf{s}$ column vector of capacity degradations with size $\left[A \times 1\right]$, $\mathbf{s} = \left[s_1, s_2, ..., s_a\right]^T$, $\forall a \in A$.

$\mathbf{t}$ column vector of mean travel times with size $\left[A \times 1\right]$ whose element is the function of flow and capacity degradation on that link, $\mathbf{t} = \left[t_1\left(x_1, s_1\right), t_2\left(x_2, s_2\right), ..., t_a\left(x_a, s_a\right)\right]^T$, $\forall a \in A$.

$\mathbf{c}^{rs}$ column vector of mean travel costs of the paths connecting the OD pair rs. This vector with size $\left[K^{rs} \times 1\right]$ can be derived from $\mathbf{c}^{rs} = \left(\mathbf{\Delta}^{rs}\right)^T \cdot \rho \cdot \mathbf{t}$ where ρ is the value time factor.

In this paper we assume that the link travel time functions are single valued and continuously differentiable. In addition, Jacobian $\nabla_{\mathbf{x}} \mathbf{t}$ is positive definite.

2.2 Stochastic user equilibrium model

In reality, travellers may not know precisely, nor perceive identically, the travel times or costs they will experience on their journeys. This condition implies some forms of stochastic model. In this paper, we assume the travellers' route choice behaviour to follow probit-based stochastic user equilibrium (SUE) (Daganzo and Sheffi, 1977). The travel time function of a degradable link is assumed to follow a standard Bureau of Public Roads (BPR) function as

$$t_a\left(x_a, s_a\right) = t_a^0 + b_a \left(\frac{x_a}{y_a - s_a}\right)^{n_a}, \quad \forall a \in A, \tag{1}$$

where t_a^0 is the free-flow travel time; b_a and n are calibration parameters; and s_a is the value of link capacity degradation.

We assume that travellers only consider personal travel cost in the disutility of their trips. Then, the perceived travel cost using path k is expressed as

$$\tilde{c}_k^{rs} = c_k^{rs} + \varepsilon_k^{rs}, \quad \forall k \in K^{rs}, rs \in \Omega, \tag{2}$$

where c_k^{rs} is the mean path travel cost, i.e. each element of $\mathbf{c}^{rs}$, and ε_k^{rs} is the perception error of path travel cost.

For the probit-based SUE, the vector of travel cost perception errors of the paths connecting OD pair rs, i.e. $\boldsymbol{\varepsilon}^{rs}$, is assumed to follow a Multivariate Normal distribution with zero mean and variance-covariance matrix $\mathbf{\Sigma}^{rs}$. We also assume that the stochastic components of $\mathbf{\Sigma}^{rs}$ have a non-degenerate joint probability density function that is continuous and strictly positive. In addition, $\mathbf{\Sigma}^{rs}$ must be dependent on $\mathbf{c}^{rs}$. These assumptions are useful when we derive the sensitivity expression, which requires the inverse of $\mathbf{\Sigma}^{rs}$. If $\mathbf{\Sigma}^{rs}$ is singular, $\mathbf{c}^{rs}$ cannot be invertible. To deal with this problem, Connors *et al*. (2007) suggested a process for constructing the path set in advance to avoid any rank deficiencies in the link-path incident matrix.

The commonly used assumption is to calculate $\mathbf{\Sigma}^{rs}$ from link cost components with the joint distribution of link cost error components. This assumption also allows link cost error correlations to be specified. For simplicity, the correlations can be neglected and the link cost error can be defined as a normal distribution with zero mean and variance σ_a^2. Then, each component of $\mathbf{\Sigma}^{rs}$ can be explicitly written as

$$\mathbf{\Sigma}_{k,j}^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} \delta_{a,j}^{rs} \sigma_a^2, \quad \forall k, j \in K^{rs}, rs \in \Omega . \tag{3}$$

Thus, the join distribution of error terms e_k^{rs}, as expressed in (2), can be is obtained. The travellers of the OD pair rs choosing the k^{th} path are those who perceive to minimize their travel cost, given the current mean path cost vector $\mathbf{c}^{rs}$. The corresponding path choice probability is derived from

$$\begin{aligned} P_k^{rs} &= \Pr\left[\tilde{c}_k^{rs} \le \tilde{c}_j^{rs} \ \forall j \,\middle|\, \mathbf{c}^{rs}\right] \\ &= \Pr\left[\begin{array}{l} c_k^{rs}\left(t_a\left(x_a\right)\right) + e_k^{rs} \le \\ c_j^{rs}\left(t_a\left(x_a\right)\right) + e_j^{rs} \quad \forall j \in K^{rs} \end{array}\right] \end{aligned} \tag{4}$$

where $\Pr\left[\cdot\right]$ denotes the probability.

There is no closed form for solving the probit path choice probability, as defined in (4). However, several methods can be used for determining P_k^{rs}, e.g. numerical integration (Genz, 1992), Monte Carlo simulation (Sheffi and Powell, 1981), or Mendell-Elston analytic approximation (Mendell and Elston, 1974).

We assume that the path choice probabilities depend on the path travel costs of that OD pair only (they do not rely on path travel costs of other OD movements). This assumption lessens the size of probit path choice probability Jacobian ($\nabla_{\mathbf{c}}\mathbf{Pr}$) to be a block diagonal matrix by OD pair, i.e.

$$\nabla_{\mathbf{c}}\mathbf{Pr} = \begin{bmatrix} \nabla_{\mathbf{c}^1}\mathbf{Pr}^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \nabla_{\mathbf{c}^{rs}}\mathbf{Pr}^{rs} \end{bmatrix} \text{ with } \nabla_{\mathbf{c}^{rs}}\mathbf{Pr}^{rs} = \begin{bmatrix} \dfrac{\partial \mathrm{Pr}_1^{rs}}{\partial c_1^{rs}} & \cdots & \dfrac{\partial \mathrm{Pr}_1^{rs}}{\partial c_k^{rs}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \mathrm{Pr}_k^{rs}}{\partial c_1^{rs}} & \cdots & \dfrac{\partial \mathrm{Pr}_k^{rs}}{\partial c_k^{rs}} \end{bmatrix}. \tag{5}$$

At equilibrium condition, no traveller can reduce *perceived* travel cost by unilaterally changing his/her path. Following Sheffi (1985), a probit-based SUE can be formulated as the fixed-point (FP) problem as

$$\mathbf{x} = \sum_{rs \in W} \Delta^{rs} \cdot q^{rs} \cdot \mathbf{P}^{rs}\left(\mathbf{c}^{rs}\left(\mathbf{x}\right)\right), \tag{6}$$

where $\mathbf{x} = \sum_{rs \in W} \Delta^{rs} \cdot \left(q^{rs} \cdot \mathbf{P}^{rs}\right)$.

In (6), $\mathbf{c}^{rs}$ is derived from $\mathbf{t}$ and further expressed as a function of the link flow vector $\mathbf{x}$. Eq. (6) is thus the FP condition. Several methods, as used for solving the FP of equilibrium condition under fixed demand, can be found from Fisk (1980), Maher and Hughes (1997), and Sheffi (1985).

3 SENSITIVITY ANALYSIS METHOD FOR VULNERABILITY ANALYSIS

3.1 Network vulnerability measure

Various measures have been proposed for assessing network vulnerability (Taylor, 2008). For illustrative purpose, we adopt a general form of relative accessibility index and the concept of random utility to define the accessibility index on each OD, i.e. AI^{rs}, as

$$\begin{aligned} AI^{rs} &= \frac{1}{E\left[\text{mean path travel cost}^{rs}\right]} \\ &= \frac{q^{rs}}{\sum_{k \in K^{rs}} f_k^{rs} c_k^{rs}} \qquad \forall rs \in W, \end{aligned} \tag{7}$$

where $E[.]$ is the expectation operator.

The proposed accessibility index, as expressed in (7), can represent the socio-economic impact when any link within the OD pair rs is partially degraded or totally closed.
A network accessibility index (*AI*) is then derived by following a normalised form of Hansen integral accessibility index (Davidson, 1977) as

$$AI = \frac{\sum_{rs \in W} q^{rs} \cdot AI^{rs}}{\sum_{rs \in W} q^{rs}}. \tag{8}$$

3.2 Sensitivity expression of network vulnerability measure

The AI of degraded network is approximated by using the first-order Taylor series expansion, i.e.

$$AI_{\text{degraded}} \approx AI_{\text{full}} + \nabla_{\mathbf{s}} AI_{\text{full}} \cdot \left(\mathbf{s} - \mathbf{s}_0\right), \tag{9}$$

where AI_{full} and AI_{degraded} are the network accessibility indices calculated under full (normal) and degraded network conditions, respectively, and $\nabla_{\mathbf{s}} AI_{\text{full}}$ is the gradient of AI_{full} with respect to $\mathbf{s}$.

In (9) AI_{full} can be obtained by using the equilibrium flows under normal network condition as the input of (7) and (8). $\left(\mathbf{s} - \mathbf{s}_0\right)$ is the change of link capacity degradations between degraded and normal network conditions. For the normal network condition, $\mathbf{s}_0 = \mathbf{0}$ since there is no link capacity degradation or link closure under normal network condition. Once the Jacobian $\nabla_{\mathbf{s}} AI_{\mathrm{full}}$ is derived, the approximated AI_{degraded} can then be calculated. Note that the proposed method only requires a single computation of a network equilibrium assignment for calculating AI_{full}. The SA technique thus substantially reduces the computational effort as compared to the traditional approach. This approximation method may not be accurate if the travel time function is highly nonlinear. However, a higher order of Taylor series expansion can be adopted to tackle this problem.

For illustration, in this paper we assume the *n* parameter in the BPR function is equal to two for all links, i.e. $n_a = 2$, $\forall a \in A$. $\nabla_{\mathbf{s}} AI_{\mathrm{full}}$ can be formulated by using the *AI* definition, as expressed in (8), and the chain rule as

$$\nabla_{\mathbf{s}} AI_{\mathrm{full}} = \frac{\sum_{rs \in W} q^{rs} \cdot \nabla_{\mathbf{s}} \mathbf{AI}^{rs}}{\sum_{rs \in W} q^{rs}}. \tag{10}$$

In (10), the Jacobian matrix $\nabla_{\mathbf{s}} \mathbf{AI}^{rs}$ is

$$\nabla_{\mathbf{s}} \mathbf{AI}^{rs} = \begin{bmatrix} \frac{\partial A^1}{\partial s_1} & \cdots & \frac{\partial A^1}{\partial s_a} \\ \vdots & \ddots & \vdots \\ \frac{\partial A^{rs}}{\partial s_1} & \cdots & \frac{\partial A^{rs}}{\partial s_a} \end{bmatrix}, \tag{11}$$

where each row of $\nabla_{\mathbf{s}} \mathbf{AI}^{rs}$, which represents the derivatives of AI^{rs} with respect to link capacity degradations s_a, $\forall a \in A$, can be calculated from

$$\begin{aligned} \nabla_{\mathbf{s}} AI^{rs} &= \frac{-q^{rs}}{\left(\left(\mathbf{f}^{rs}\right)^T \cdot \mathbf{c}^{rs}\right)^2} \left[\left(\mathbf{f}^{rs}\right)^T \cdot \nabla_{\mathbf{s}} \mathbf{c}^{rs} + \left(\mathbf{c}^{rs}\right)^T \cdot \nabla_{\mathbf{s}} \mathbf{f}^{rs}\right] \\ &= \frac{-q^{rs}}{\left(\left(\mathbf{f}^{rs}\right)^T \cdot \mathbf{c}^{rs}\right)^2} \left[\left(\mathbf{f}^{rs}\right)^T \cdot \left(\Delta^T \cdot \nabla_{\mathbf{s}} \mathbf{t}\right) + \left(\mathbf{c}^{rs}\right)^T \cdot \left(q^{rs} \cdot \nabla_{\mathbf{s}} \mathbf{p}^{rs}\right)\right] \quad \forall rs \in W. \end{aligned} \tag{12}$$

Each diagonal element of $\nabla_{\mathbf{s}} \mathbf{t}$, as expressed in (12), can be determined from

$$\frac{\partial t_a}{\partial s_a} = n_a b_a \frac{\left(x_a\right)^{n_a}}{\left(y_a - s_a\right)^{n_a + 1}}. \tag{13}$$

Since $\partial t_a / \partial x_b = 0$ for $a \neq b$ and $b \in A$, all off-diagonal entries of $\nabla_{\mathbf{s}} \mathbf{t}$ are zero.

To complete (12), Jacobian of path choice probabilities ($\nabla_{\mathbf{s}} \mathbf{p}^{rs}$) can be derived by using the sensitivity analysis.

3.3 Jacobian of path choice probabilities

We formulate the gap function of path choice probabilities, as expressed in (4), as $\Theta(\mathbf{P},\mathbf{s}) \equiv \mathbf{P} - \mathbf{Pr}(\mathbf{P}(\mathbf{s}),\mathbf{s})$. Let $\mathbf{P}^*(\mathbf{s})$ be the solution of probit-based SUE for any given value of $\mathbf{s}$, i.e. $\Theta(\mathbf{P}^*(\mathbf{s}),\mathbf{s}) = \mathbf{0}$. Assuming that all related functions are differentiable, the first-order linear approximation of $\Theta(\mathbf{P},\mathbf{s})$, evaluated around the point $(\mathbf{P}^*(\mathbf{s}_0),\mathbf{s}_0)$, can be formulated as $\Theta(\mathbf{P},\mathbf{s}) \approx \Theta(\mathbf{P}^*(\mathbf{s}_0),\mathbf{s}_0) + \nabla_{\mathbf{P}} \Theta\big|_{\mathbf{P}^*(\mathbf{s}_0)\mathbf{s}_0} (\mathbf{P} - \mathbf{P}^*(\mathbf{s}_0)) + \nabla_{\mathbf{s}} \Theta\big|_{\mathbf{P}^*(\mathbf{s}_0)\mathbf{s}_0} (\mathbf{s} - \mathbf{s}_0)$ where $\nabla_{\mathbf{P}} \Theta\big|_{\mathbf{P}^*(\mathbf{s}_0)\mathbf{s}_0}$ and $\nabla_{\mathbf{s}} \Theta\big|_{\mathbf{P}^*(\mathbf{s}_0)\mathbf{s}_0}$ are the Jacobian matrices (denoted as $\mathbf{J}_1$ and $\mathbf{J}_2$) of $\Theta(\mathbf{P},\mathbf{s})$ with respect to $\mathbf{P}$ and $\mathbf{s}$, respectively, evaluated at the solution $(\mathbf{P}^*(\mathbf{s}_0),\mathbf{s}_0)$. At equilibrium condition, the approximation of the gap function becomes $\mathbf{0} \approx \mathbf{0} + \mathbf{J}_1(\mathbf{P} - \mathbf{P}^*(\mathbf{s}_0)) + \mathbf{J}_2(\mathbf{s} - \mathbf{s}_0)$ and hence

$$\begin{aligned} \nabla_{\mathbf{s}} \mathbf{P}^{rs} &= \lim_{\mathbf{s} \to \mathbf{s}_0} \frac{(\mathbf{P} - \mathbf{P}^*(\mathbf{s}_0))}{(\mathbf{s} - \mathbf{s}_0)} \\ &= -\mathbf{J}_1^{-1} \mathbf{J}_2, \end{aligned} \tag{14}$$

with

$$\mathbf{J}_1 = \mathbf{I}_{[K^{rs} \times K^{rs}]} - \nabla_{\mathbf{c}^{rs}} \mathbf{Pr}^{rs} \cdot \left[(\Delta^{rs})^T \cdot (\nabla_{\mathbf{x}} \mathbf{t} \cdot \nabla_{\mathbf{P}^{rs}} \mathbf{x}) \right], \tag{15}$$

$$\mathbf{J}_2 = -\nabla_{\mathbf{c}^{rs}} \mathbf{Pr}^{rs} \cdot \left[(\Delta^{rs})^T \cdot \nabla_{\mathbf{s}} \mathbf{t} \right], \tag{16}$$

where **I** is the identity matrix with size $[K^{rs} \times K^{rs}]$.

In (15) and (16) the calculation of probit path choice Jacobian ($\nabla_{\mathbf{c}^{rs}} \mathbf{Pr}^{rs}$), which are the derivatives of the probit path choice probabilities with respect to path travel costs for each OD pair rs, is clearly explained in Clark and Watling (2002). In this paper we use a Monte Carlo simulation for solving the choice probabilities P_k^{rs}, as written in (4).

In (15) $\nabla_{\mathbf{x}} \mathbf{t}$ can be derived from (13), whereas $\nabla_{\mathbf{P}^{rs}} \mathbf{x}$ is calculated from

$$\nabla_{\mathbf{P}^{rs}}\mathbf{x} = \mathbf{\Delta}^{rs} \cdot \left(q^{rs} \cdot \mathbf{I}_{\left[K^{rs} \times K^{rs} \right]} \right). \tag{17}$$

Finally, each diagonal element of $\nabla_{\mathbf{s}}\mathbf{t}$, as expressed in (16), is determined from

$$\left. \frac{\partial t_a}{\partial s_a} \right|_{s_0} = \nu_a \beta_a \frac{\left(x_a^* \right)^{\nu_a}}{\left(y_a - s_0 \right)^{\nu_a + 1}}, \tag{18}$$

where off-diagonal entries of $\nabla_{\mathbf{s}}\mathbf{t}$ are all zero.

In summary, all derivatives, as defined from (15) to (18), can be substituted in the reverse order to attain the Jacobian of path choice probabilities, as expressed in (14). Then, the corresponding Jacobian can be substituted into (12) to achieve $\nabla_{\mathbf{s}} AI^{rs}$.

3.4 Critical link identification

The absolute and relative changes of accessibility indices under full and degraded network conditions can be assessed, respectively, from

$$\Delta AI = AI_{\text{full}} - AI_{\text{degraded}}, \tag{19}$$

$$RAI = 1 - \frac{AI_{\text{degraded}}}{AI_{\text{full}}}. \tag{20}$$

By substituting AI_{degraded}, as defined in (9), to (19) and (20), the absolute and relative changes of *AI* are rewritten as

$$\Delta AI = -\nabla_{\mathbf{s}} AI_{\text{full}} \cdot \mathbf{s}, \tag{21}$$

$$RAI = -\frac{\nabla_{\mathbf{s}} AI_{\text{full}} \cdot \mathbf{s}}{AI_{\text{full}}}. \tag{22}$$

Vulnerable links can then be identified by ranking the links according to ΔAI (or RAI). The link with a higher value of ΔAI (or RAI) is more critical. Note that the proposed technique can also be applied to other vulnerability measures.

4 NUMERICAL EXAMPLES

This section demonstrates the feasibility and efficiency of the proposed method by testing with the networks of Sioux Falls city and Bangkok metropolitan area. We aim to compare the results of the proposed SA technique with those from the traditional approach. For all tests, we used a personal computer with Intel Core 2 Duo 1.86 GHz CPU, 4 GB RAM, and Windows 7 Ultimate as the operating system (OS). For the case with the traditional approach, SATURN was used for solving the probit-based SUE under both normal and degraded network conditions. The parameter KOB for the probit model in SATURN is set at 2 so that the link travel times follow a normal distribution with mean t_a and variance $= SUET \times t_a$. The *SUET*, which represents the coefficient of link travel time error, is assumed to be 0.3. Since the variance of a normal distribution is additive (not the standard deviation), the variance of link travel time is thus chosen for the probit model. The probit-based SUE problem is solved with a maximum number of iterations of 500 and a gap tolerance of 10E-8.

For the case with the SA approach, once the equilibrium flows are calculated under the normal network condition, the approximated ΔAI is then determined by using (21). All explicit sensitivity expressions in the previous section allow the implementation of the proposed method in any matrix-based mathematical languages. In this paper, we use MATLAB. The details of all tests are as follows.

4.1 Sioux Falls network

The first test is with a middle-size network of Sioux Falls city as shown in Figure 1. The network consists of 24 zones, 76 road links, and 528 OD pairs. The link travel time function is assumed to be $t_a = t_a^0 + b_a \left[x_a / \left(y_a - s_a \right) \right]^2, \ \forall a \in A$. The OD travel demands and parameters of the travel time function are the same as Suwansirikul *et al*. (1987) and Bar-Gera (2001).

For the case with the traditional approach, each link is closed in turn in which the performance of the remaining network is tested by carrying out the SUE assignment with SATURN. Each of them represents the disruption on each link in the network. Thus, the traditional approach involves running the Probit SUE assignments for 76 times. In addition, we also consider four different levels of link capacity degradations, 25%, 50%, 75% and 100% (link closure), and three conditions of network congestions, including free flow, intermediate flow, and highly congested flow. Thus, 76x4x3 + 1 (normal network condition) = 912 test networks are developed in order to assess the AIs under different network conditions.

The traffic assignment problems of all 912 test networks are solved. The values of AI change, i.e. ΔAI, for the traditional approach are then calculated by using (19). Note that *RAI* can also be used for identifying critical links.

For the case with the SA approach, we follow the suggestion of Connors *et al*. (2007) by generating a total of 1,306 paths (for the Sioux Falls network) in advance to avoid any rank deficiencies in the link-path incident matrix for each OD movement, which will cause a non-invertible $\mathbf{\Sigma}^{rs}$. These paths are generated (for calculating the path covariance in the probit model) by assuming that the travel demands are increased by 10 times of the original demands. The approximated values of $\mathrm{D}AI$ under different conditions are calculated by using (21) and (10).

Next, critical links for each approach are ranked in accordance to the corresponding changes of AI. In this paper, Spearman's rho r (Spearman, 1904) is used to evaluate the correlation between the two sets of critical link ranks. Spearman's rho is used here because it does not require a linear relationship of two sets of rankings, which is the case in Pearson's correlation coefficient (Pearson, 1920). In addition, this rank correlation coefficient, which is a non-parametric measure, involves low computational effort.

Figure 2 shows the results of r as evaluated by comparing the critical link rankings between SA and traditional approaches. Under small levels of capacity degradation and network congestion, the ranks from both methods are similar, which give the maximum r of 0.9923. When the capacity degradation increases, the correlation decreases. For the case with 100% capacity degradation (i.e. link closure), the r is 0.68, 0.59, and 0.43 under the low, medium, and high level of congestion respectively. Note that the impact of the increase in network congestion is similar to that of the increase in capacity degradation.

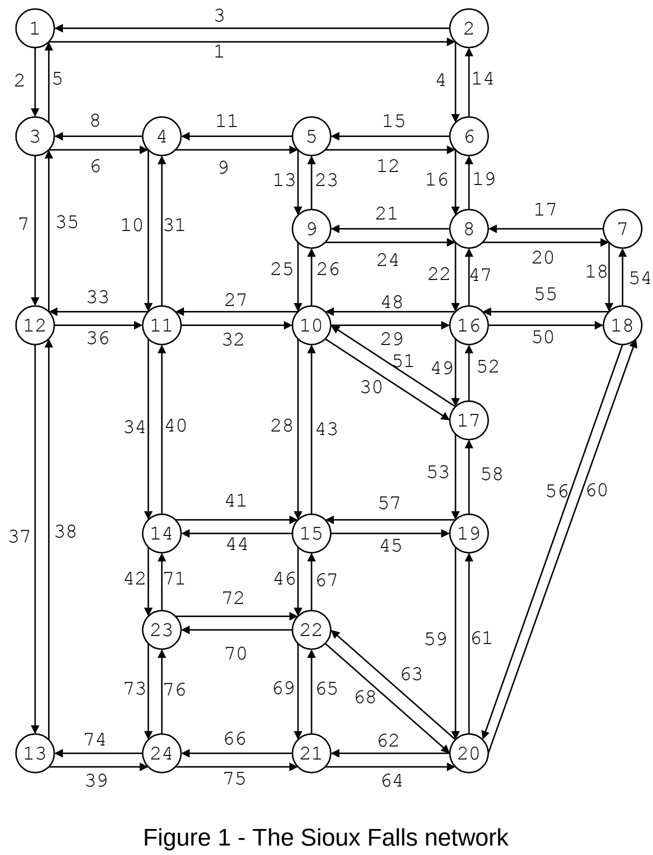


Figure 1 - The Sioux Falls network

From the results, the SA method may not exactly match the rank of critical links obtained from the traditional approach. The accuracy of the proposed method can be improved by introducing the error and a higher-order in the approximation. However, for the Sioux Falls network the SA method can reduce the computational time to about 310 times lower than the traditional approach. The other advantage of the proposed method is that it does not require large storage for generating the degraded network files, which is worse for a large-scale road network.

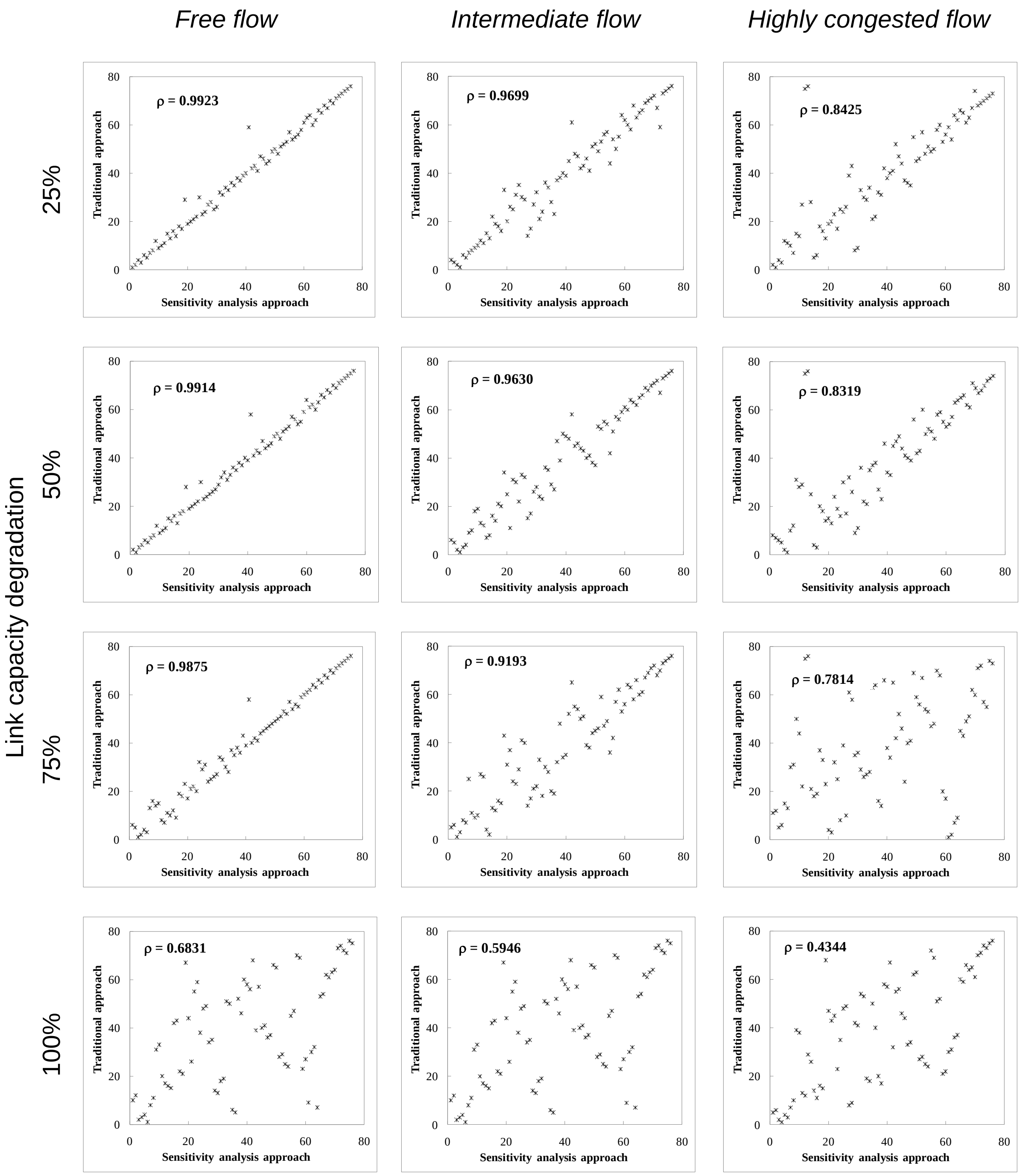


Figure 2 - Rank correlation coefficients between SA and traditional approaches

4.2 Bangkok metropolitan area network

Bangkok metropolitan area (BMA) includes Bangkok and five surrounding provinces: Nonthaburi, Samut Prakan, Pathum Thani, Samut Sakhon and Nakhon Pathom. It covers an area of 7,762 km^2 and approximately has a population of 9,014,470 as of December 31, 2008 (DOPA, 2009). The BMA network, including the national highways and major arterial roads, is shown in Figure 3. The BMA network consists of 243 zones, 4,598 road links, and 59,049 OD pairs. Total travel demand is about 1,057,717 passenger car equivalent units (pcu) per hour during normal condition. Similar to the case of Sioux Falls network, we generate a total of 61,593 paths for calculating the path covariance in the probit model.

For the case with the traditional approach, we generate the degraded network cases, solve the assignment problem for each case, and then calculate the values of AI. Three levels of capacity degradations, including 25%, 50% and 100%, are considered for BMA network. The values of $\mathrm{D}AI$ are calculated and used for identifying the critical links. The Spearman's r is then determined for two sets of critical link rankings. Figure 4 illustrates the ranks of the critical links obtained from two approaches under 25% of capacity degradations. The rank correlation between SA and traditional approach is quite low, i.e. $r = 0.3117$. However, road network manager(s) or transport planner(s) may only want to know several top critical links, rather than all critical links. Figure 5 shows the percentage of critical links obtained from SA method which are included in the set of the critical links ranked by the traditional approach.

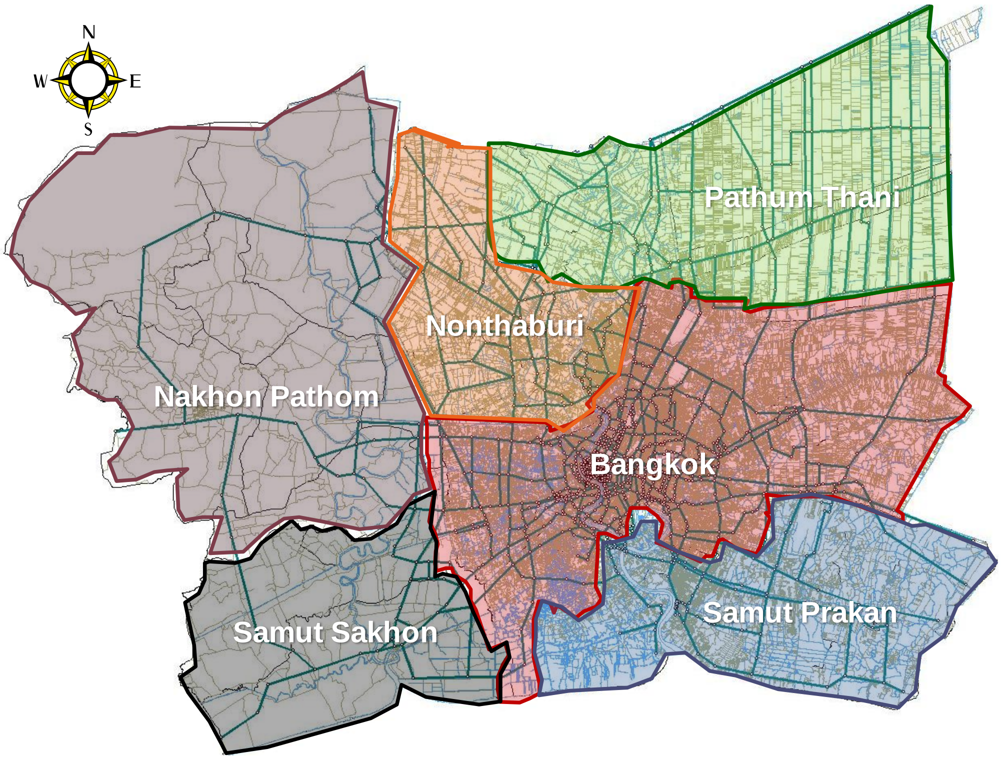


Figure 3 - Bangkok metropolitan area network

As shown in Figure 4, the SA method does not provide the exact ranks of critical links, compared to the traditional approach. However, we can apply the SA method to select a subset of critical links and use the traditional approach to evaluate the actual critical ranks of those selected links. From Figure 5, if the planner specifies a confidence level of 75%, then the top 2,000 critical links obtained from the SA method are chosen and re-ranked. With this short-list approach, the proposed scheme can decrease the computational time from 14.4 days (traditional approach) to about 1 day (SA method). This result shows the feasibility and efficiency of the SA method for the large-scale networks. In addition, the computational time can be reduced if a smaller level of confidence is considered. By doing so, the top 50 critical links of BMA network under 25%, 50% and 100% of capacity degradations are demonstrated in Figure 6, Figure 7, and Figure 8, respectively. The top critical links are as follows:

- Highways No. 321, section Muang Nakhon Pathom to Mung Suphan Buri
- Highways No. 346, section Kamphaeng Saen to Sai Noi
- Highways No. 4, section Sanam Chan to Phra Pathom Chedi
- Highways No. 325, section Damnoen Saduak to Muang Samut Songkham
- Highways No. 37 (south corridor road), section Tha Kham to Bang Chak
- Highways No. 3, section Muang Samut Prakan to Pak Nam
- Highways No. 3344 (Srinagarindra road) middle east corridor road
- Near the entrance of Highways No. 9 (eastern outer ring road), section Saphan Sung to Watcharapol.

The top critical links identified in this paper are similar to those in reality since these are links connecting between major activities and serve high level of travel demands. In addition, most of these links do not have alternative paths.

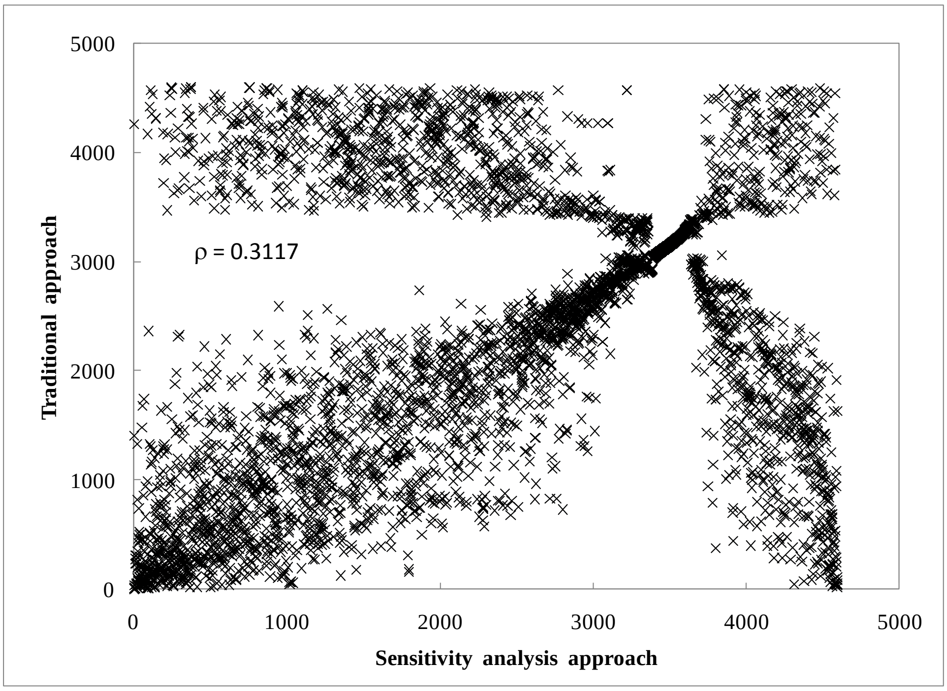


Figure 4 - Correlation of link ranks from sensitivity analysis and traditional approaches

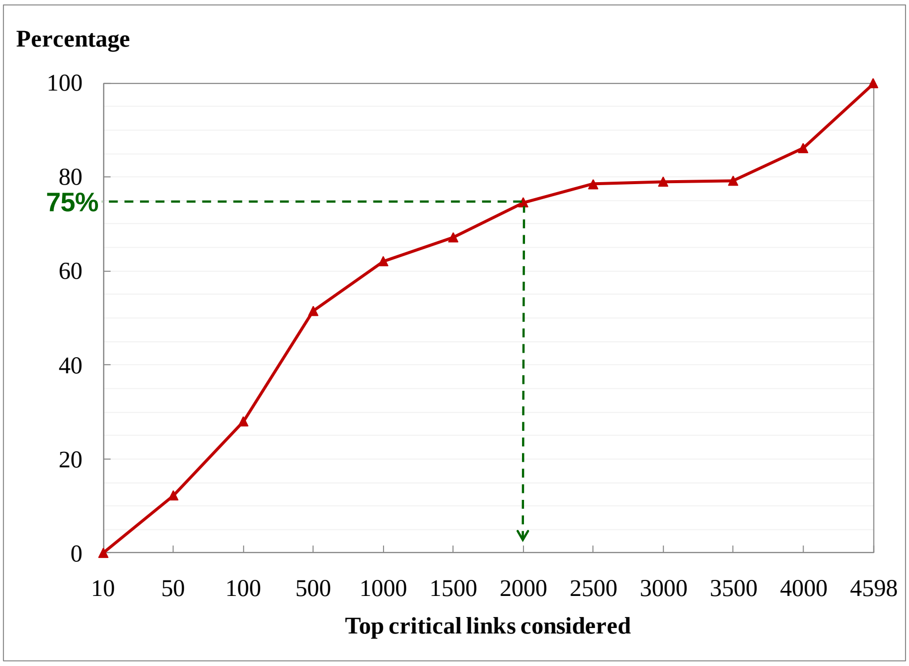


Figure 5 - Probabilities that SA links are included in the top critical links

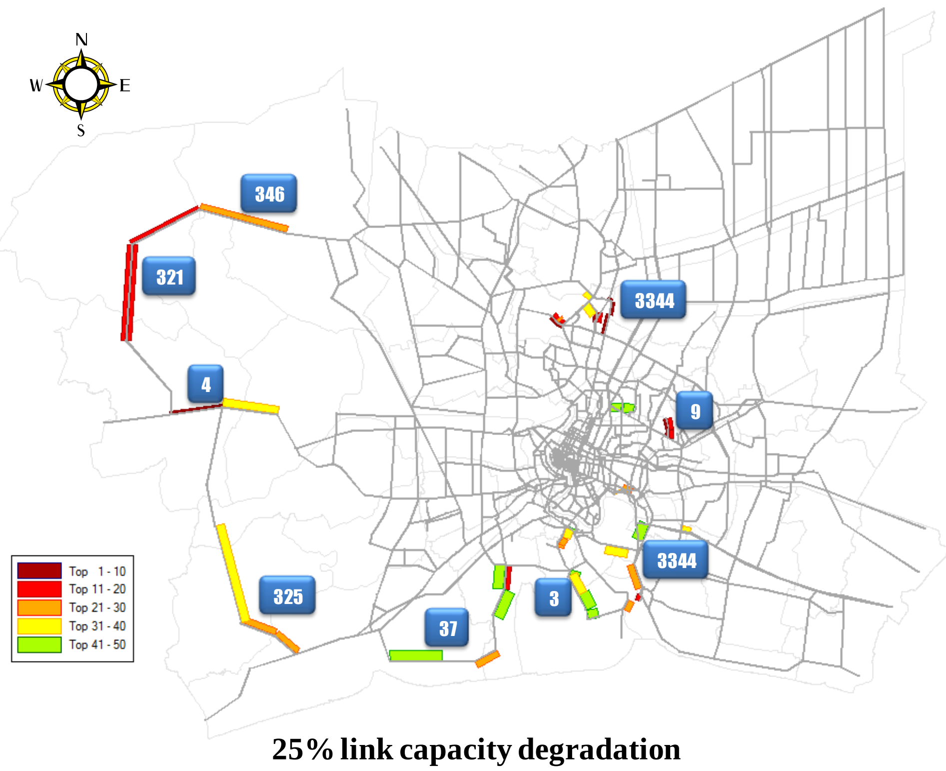
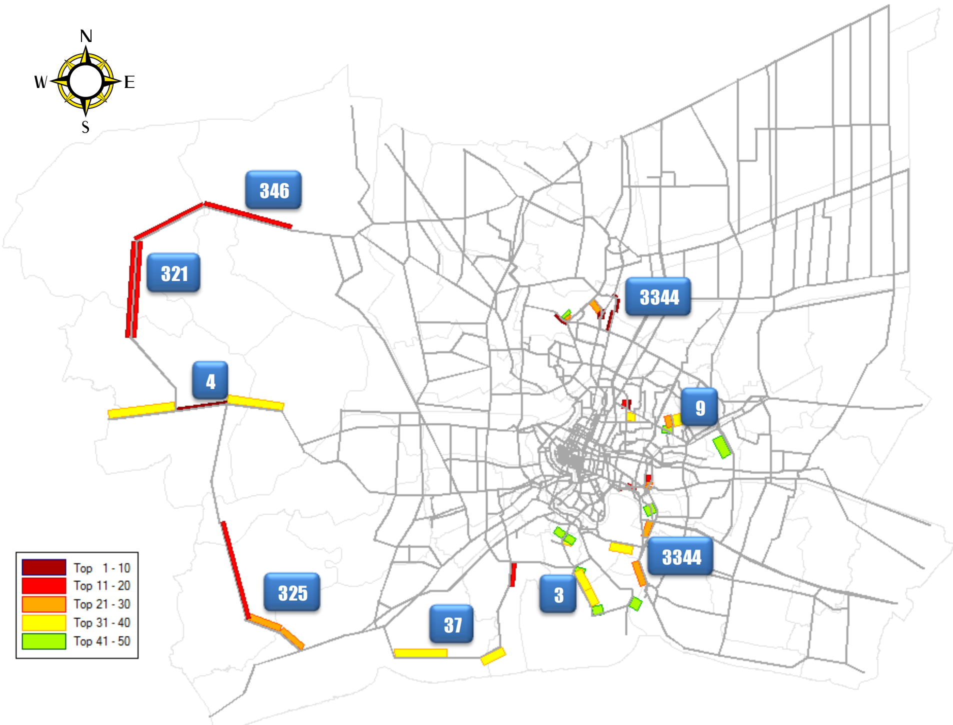
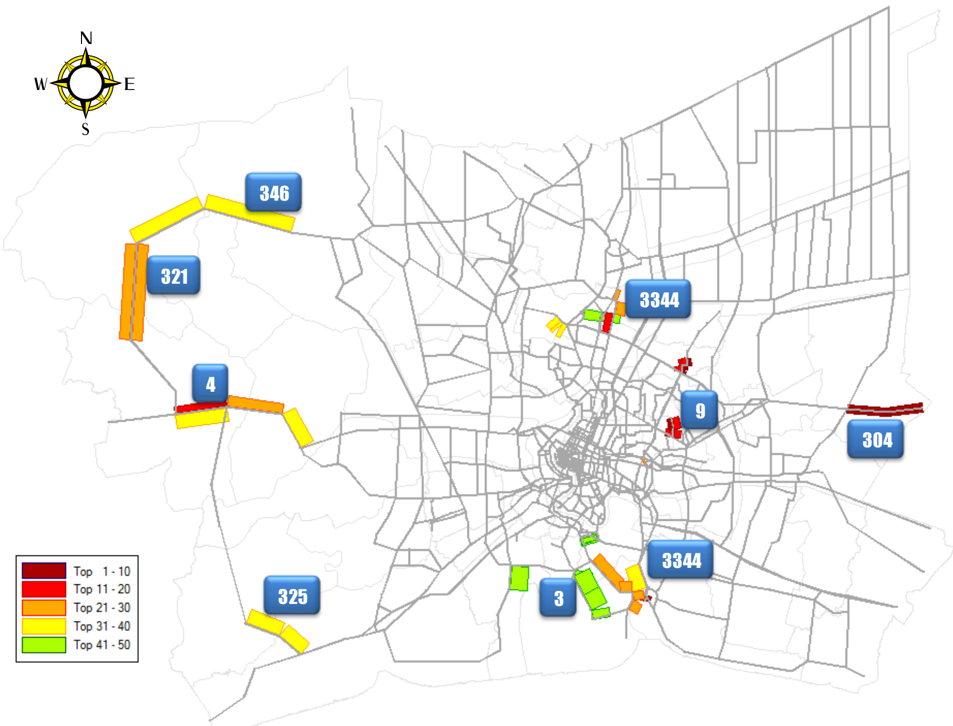


Figure 6 - Critical link identification under 25% degradation



50% link capacity degradation

Figure 7 - Critical link identification under 50% degradation



100% link capacity degradation

Figure 8 - Critical link identification under 100% degradation

5 CONCLUSIONS AND DISCUSSION

In this paper, a sensitivity analysis based approach was proposed to improve the computational efficiency and allow for the large-scale application of network vulnerability analysis. For illustration, a network accessibility index (AI) was introduced by following a normalised form of Hansen integral accessibility index to assess the network vulnerability. The approximated AI was formulated by using the first-order Taylor expansion. All sensitivity expressions were also derived and given. Under network disruption, links with a large change of AI were considered more critical. Critical links were then identified by ranking the links according to the change of AI. The proposed method was tested with the Sioux Falls and Bangkok metropolitan area networks. The results demonstrated the feasibility of the proposed technique to the application of large-scale road networks. The SA method also did improve the efficiency in terms of computational time and required memory storage, compared to the traditional approach. The accuracy of critical ranking is an issue to improve the efficiency of the SA approach. The proposed method can also be extended to consider other vulnerability measures and the cases of multiple user-classes and multimodal network.

ACKNOWLEDGEMENTS

The first author would like to thank the Hong Kong Polytechnic University for providing the postgraduate research studentship. The research is also supported by a General Research Fund project of Hong Kong Research Grant Council (PolyU5271/08E and).

REFERENCES

Asakura, Y. and M. Kashiwadani (1991). Road network reliability caused by daily fluctuation of traffic flow. Proceedings of the 19th PTRC Summer Annual Meeting (Seminar G), 19, 73-84.

Bar-Gera, H. (2001). Transportation Network Test Problems.

Bell, M. G. H. and Y. Iida (1997). Transportation network analysis. J. Wiley, New York.

Berdica, K. (2002). An introduction to road vulnerability: what has been done, is done and should be done. Transport Policy, 9, 117-127.

Berdica, K. and L. Mattsson (2007). Vulnerability: a model-based case study of the road network in Stockholm. In: Critical Infrastructure (M. M. Fischer et al., ed.), pp. 81-106. Springer, New York.

Chen, A., C. Yang, S. Kongsomsaksakul and M. Lee (2007). Network-based accessibility measures for vulnerability analysis of degradable transportation networks. Networks and Spatial Economics, 7, 241-256.

Chen, A., H. Yang, H. K. Lo and W. H. Tang (1999). Capacity related reliability for transportation networks. Journal of Advanced Transportation, 33, 183-200.

Chen, A., H. Yang, H. K. Lo and W. H. Tang (2002). Capacity reliability of a road network: an assessment methodology and numerical results. Transportation Research B, 36, 225-252.

Chow, A. H. F. and H. K. Lo (2007). Sensitivity analysis of signal control with physical queuing: delay derivatives and an application. Transportation Research B, 41, 462-477.

Clark, S. D. and D. Watling (2002). Sensitivity analysis of the probit-based stochastic user equilibrium assignment model. Transportation Research B, 36, 617-635.

Connors, R. D., A. Sumalee and D. Watling (2007). Sensitivity analysis of the variable demand probit stochastic user equilibrium with multiple user-classes. Transportation Research B, 41, 593-615.

D'Este, G. M. and M. A. P. Taylor (2003). Network vulnerability: an approach to reliability analysis at the level of national strategic transport networks. In: The network reliability of transport. (M. G. H. Bell and Y. Iida, ed.), pp. 23-44. Pergamon, Oxford.

Daganzo, C. F. and Y. Sheffi (1977). On stochastic models of traffic assignment. Transportation Science, 11, 253.

Davidson, K. B. (1977). Accessibility in transport/land-use modelling and assessment. Environment and Planning A, 9, 1401-1416.

Davis, G. A. (1994). Exact local solution of the continuous network design problem via stochastic user equilibrium assignment. Transportation Research B, 28, 61-75.

DOPA (2009). Department of provincial administration. Population statistics (in Thai).

Du, Z. P. and A. Nicholson (1997). Degradable transportation systems: sensitivity and reliability analysis. Transportation Research B, 31, 225-237.

Fisk, C. (1980). Some developments in equilibrium traffic assignment. Transportation Research B, 14, 243-255.

Genz, A. (1992). Numerical computation of multivariate normal probabilities. J. Computational and Graphical Statistics, 1, 141-149.

Hansen, W. G. (1959). How accessibility shapes land use. J. the American Institute of Planners, 25, 73 - 76.

Jenelius, E. (2009). Network structure and travel patterns: explaining the geographical disparities of road network vulnerability. J. Transport Geography, 17, 234-244.

Jenelius, E., T. Petersen and L. MATTSSON (2006). Importance and exposure in road network vulnerability analysis. Transportation Research A, 40, 537-560.

Leurent, F. (1998). Sensitivity and error analysis of the dual criteria traffic assignment model. Transportation Research B, 32, 189-204.

Lleras-Echeverri, G. and M. Sanchez-Silva (2001). Vulnerability analysis of highway networks, methodology and case study. Transport, 147, 223-230.

Magnanti, T. L. and T. R. Wong (1984). Network design and transportation planning: models and algorithms. Transportation Science, 18, 1.

Maher, M. J. and P. C. Hughes (1997). A probit-based stochastic user equilibrium assignment model. Transportation Research B, 31, 341-355.

Mendell, N. R. and R. C. Elston (1974). Multifactorial qualitative traits: genetic analysis and prediction of recurrence risks. Biometrics, 30, 41-57.

Nakayama, S., R. D. Connors and D. Watling (2009). Estimation of parameters of network equilibrium models: a maximum likelihood method and statistical properties of Network Flow. In: Transportation and Traffic Theory 2009: Golden Jubilee (W. H. K. Lam et al., ed.), pp. 39-56. Springer, New York.

Patriksson, M. and R. T. Rockafellar (2002). A mathematical model and descent algorithm for bilevel traffic management. Transportation Science, 36, 271-291.

Pearson, K. (1920). Notes on the history of correlation. Biometrika, 13, 25-45.

Robinson, S. M. (2006). Strong regularity and the sensitivity analysis of traffic equilibria: a comment. Transportation Science, 40, 540-542.

Sheffi, Y. (1985). Urban transportation networks: equilibrium analysis with mathematical programming methods, Prentice-Hall, New Jersey.

Sheffi, Y. and W. Powell (1981). A comparison of stochastic and deterministic traffic assignment over congested networks. Transportation Research B, 15, 53-64.

Spearman, C. (1904). The proof and measurement of association between two things. The American Journal of Psychology, 100, 441-471.

Sumalee, A., R. D. Connors and D. Watling (2006a). An optimal toll design problem with improved behavioural equilibrium model: the case of the probit model. In: Mathematical and Computational Models for Congestion Charging (S. Lawphongpanich et al., ed.), 219-240.

Sumalee, A. and F. Kurauchi (2006). Network capacity reliability analysis considering traffic regulation after a major disaster. Networks and Spatial Economics, 6, 205-219.

Sumalee, A., P. Luathep, W. H. K. Lam, and R. D. Connors (2009). Evaluation and design of transport network capacity under demand uncertainty. Transportation Research Record, 2090, 17-28.

Sumalee, A., D. Watling and S. Nakayama (2006b). Reliable network design problem: case with uncertain demand and total travel time reliability. Transportation Research Record, 1964, 81-90.

Suwansirikul, C., T. L. Friesz and R. L. Tobin (1987). Equilibrium decomposed optimization: a heuristic for the continuous equilibrium network design problems. Transportation Science, 21, 254-263.

Taylor, M. A. P. and G. M. D'Este (2007). Transport network vulnerability: a method for diagnosis of critical locations in transport infrastructure systems. In: Critical Infrastructure (M. M. Fischer et al., ed.), pp. 9-30. Springer, New York.

Taylor, M., S. Sekhar and G. M. D'Este (2006). Application of accessibility based methods for vulnerability analysis of strategic road networks. Networks and Spatial Economics, 6, 267-291.

Taylor, M. A. P. (2008). Critical transport infrastructure in urban areas: impacts of traffic incidents assessed using accessibility-based network vulnerability analysis. Growth and Change, 39, 593-616.

Wakabayashi, H. And Y. Iida (1992). Upper and lower bounds of terminal reliability of road networks: an efficient method with boolean algebra. J. Natural disaster science, 14, 29-44.

Yang, H. and M. G. H. Bell (1997). Traffic restraint, road pricing and network equilibrium. Transportation Research B, 31, 303-314.

Yang, H., T. Sasaki, Y. Iida and Y. Asakura (1992). Estimation of origin-destination matrices from link traffic counts on congested networks. Transportation Research Part B, 26, 417-434.

Ying, J. Q. And T. Miyagi (2001). Sensitivity analysis for stochastic user equilibrium network flows-a dual approach. Transportation Science, 35, 124-133.