A DYNAMIC OPERATIONAL URBAN ECONOMIC MODEL BASED ON AGENT'S BEHAVIOR

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ABSTRACT

The real estate and land use system can be described as a chain of three sub-markets: building development, real estate ownership and residential allocation. This paper develops an economic model of the real estate ownership market, where developers offer new options to investors whom decide what options to buy –differentiated by location and real estate type- and which to sell in a pure exchange and competitive economy. A logit model represents the agents' idiosyncratic differences in their maximizing behavior of the long-term present value of their investments. We prove the market equilibrium exists and is unique under reasonable conditions, and the solution is obtained using a fixed-point algorithm. The model performance is illustrated with simulations of the equilibrium under several scenarios.

Keywords: Real estate market, equilibrium, logit

1. INTRODUCTION

The structure of the real estate market has been described and analyzed by Martínez and Roy (2004), where it is segmented into three submarkets: (i) stock production, (ii) stock ownership, and (iii) stock usage; these three submarkets are interlinked by prices and stocks, the latter differentiated by locations and building types. This paper aims to formulate the dynamics in the stock ownership's market in the context of classic urban economics.

Dynamic processes have been traditionally modeled in residential location using a microsimulation approach. This approach allows to model location in great detail and, given its flexible rule based formulation, allows the representation of complex dynamic phenomena. Among these models family, the dynamic model UrbanSim (Waddell, 2001) is an example, where individual agents are considered by Montecarlo selection and their behavior micro-simulated as choosing their locations based on exogenous probabilities and rules that cannot be violated. An observation to this class of models is that they do not consider the existence of auction strategies nor market equilibrium prices. The PECAS model (Hunt and Abraham, 2007) makes an improvement integrating simulation processes with market equilibrium conditions based on a input-output matrix structure, including goods, services, jobs, land and market clearing with exchange prices. In this case the model becomes highly complex and no proves of equilibrium existence and their stability is given.

On the other hand, with the bid-choice model, Martínez (1992) takes the urban economics discussion (Alonso, 1964; Anas, 1982), providing a solid micro-economic base to model land use, called the bid-choice model, which is applied in the operative models MUSSA and Cube Land (Martínez and Donoso, 2001). This approach is economically consistent under a static perspective and stable solutions are warranted, but it does not take into account the dynamics in the urban development process; it only models the stock usage stage of market. Martínez and Hurtubia (2006) modeled the real estate stock production submarket, introducing the delay of the construction sector and define a market's equilibrium between developers and residents or final users, considering a microeconomic bid-choice equilibrium approach at each period given real estate stock built in the past. This model, however, skips the stock ownership intermediate submarket.

In this paper we formulate a model of the ownership market. The objective is twofold: i) introduce the role of the stocks and the financial sector in the equilibrium dynamics in the global real estate-land use market, where real estate are traded as capital assets with a long term view, and where the macro-economics (interest rates and capital stocks) play a relevant role influencing residential prices; ii) identify the owners of residences and their distribution in population, whose behavior in the land use market is different than renters.

A stochastic dynamic model of the real estates' ownership market is proposed, under a market equilibrium approach to derive the price signals of the market process. The agents represent investors who maximize the present value of their profits. Since this market is understood as an intermediate market, placed in between the stock production and the residential location submarkets, there is neither production nor final consumers; it follows that it is an exchange economy. We consider a closed market where agents can only invest among the given real estate options and in the financial market, a fixed profitability option called the *bank*. This setting leads to a dynamic where at each period agents decide how to optimally distribute their capital, comparing the expected profits from leases and sales of the options in the future periods and the expected interests of investing in the bank.

It is assumed that each period *t* is divided into two sub periods such as: $t = \{t_a, t_b\}$. In the first period (t_a) all agents decide how many new shares of each real estate option he/she should buy and how many sell from their current stock; this period is considered as almost instantaneous and agents update their stock. The location and the building type of the dwelling differentiate the stock options. In the second period (t_b) all agents offer their stock in the market for rent during this period, and they themselves decide their residential location and the expenditure on other goods and services. Then, t_a is the investment period and t_b the operation period; this model focuses on the decisions that take place in the investment period, the operation period is modeled by the land use equilibrium model which is assumed exogenous in our model; hence, rents are given and they are thought to be taken from the RB&SM approach (Martínez and Henríquez, 2007). A myopic assumption is considered so agents have to make estimates of future rents and prices, such that agents consider the historic information of the real estate market and make their own predictions about future prices. Finally, we assume that the real estate supply it is also exogenous and it is obtained from the dynamic model proposed by Martínez and Hurtubia (2006).

2. MODEL VARIABLES

This market is defined by investor agents who buy and sell real estate options, in order to maximize the expected value of their profit (Π_h^t). On each period they evaluate updating their real estate stock by selling it out and deciding how to distribute their capital among all available options: real estate defined by location and type, and a default of investing in the financial market (which will be called option *bank*). These agents seek to capture net rents (after maintenance costs, c_i^t) and capital gain from their stock.

Given the complexity of the problem, it is necessary to define a large set of variables and indices for the model. Further detailed explanations will be given as the model is presented.

h, i, t: Category indices of agent, real estate option type and time period respectively, where $h \in H$ and $i \in I$. The time period may be considered one year.

S_{hi}^t: Stock of agent *h*, of option *i,* in period *t*, defined as shares of real estate units. That said, $S_{hi}^t \in \mathbb{R}^+$.

p_i^t: Sale price of option *i*, in period *t*. $p_i^t \in \mathbb{R}^+$.

r_i^t: Rent of option *i*, in period *t*. $r_i^t \in \mathbb{R}^+$.

$\bar{r}_i^t$: Net rent of option *i*, in period *t*. This variable is incorporated in order to represent the agent's decision whether to lease their properties or not, depending if the money offered by the tenants is higher than the maintenance costs, c_i^t. Then, the net rent $\bar{r}_i^t \in \mathbb{R}^+$ is defined by:

$$\bar{r}_i^t = \begin{cases} r_i^t - c_i^t & \text{si}\, r_i^t > c_i^t \\ 0 & \sim \end{cases} \qquad (2.1)$$

G_{h0}^t: Expenditure in option *bank* incurred by agent *h,* in period *t*; $G_{h0}^t \in \mathbb{R}^+$.

G_{hi}^t: Expenditure in option *i* incurred by agent *h*, in period *t*. There is a relationship between expenditure (monetary units) and the stock (quantity units) on each option *i* given by the option's selling price p_i^t. Then, the expenditure incurred by agent *h* in option *i* on period *t*, $G_{hi}^t \in \mathbb{R}^+$, is given by:

$$G_{hi}^t = \frac{S_{hi}^t}{p_i^t} \qquad (2.2)$$

O_i^t: Total supply of option *i*, in period *t*. $O_i^t \in \mathbb{R}^+$.

N_h^t: Number of investor agents class *h* at period *t*. $N_h^t \in \mathbb{R}^+$.

$\hat{y}_h^t$: Available income of agent *h*, in period *t*. It corresponds to the annual salary (y_h^t) minus location costs (r_{loc}^t) and the expenditure in other goods ($(p \cdot x_h)^t$). This way, $\hat{y}_h^t = y_h^t - [r_{loc}^t + (p \cdot x_h)^t]$.

ρ_i^t: Profit restriction factor for real estate supply. It corresponds to a cut-off function which tends to zero if selling prices approach to construction costs (or lower), and tends to one if selling prices raises high above these costs.

β_h: Discount rate of capital of agent *h*. $\beta_h \in [0,1]$.

q^t: Interest rate given by the bank in period *t*. $q^t \in \mathbb{R}^+$.

3. MODEL SPECIFICATION.

3.1. The agents' behavior.

To model the behavior of an agent, we define the profit function of agent *h* in period *t* as the income from expected leases, sales and interests, minus the incurred costs of investing in all the options (iii). This function is:

$$\Pi_h^t(\beta_h, G_i^t, q^t, \bar{r}^t, p^t, p^{t+1}) = \beta_h \Bigg[\underbrace{(1+q^t)G_{h0}^t}_{\text{(i)}} + \underbrace{\sum_{i \neq 0}\left(\frac{\bar{r}_i^t + p_i^{t+1}}{p_i^t}\right)G_{hi}^t}_{\text{(ii)}}\Bigg] - \underbrace{\Bigg[\sum_i G_{hi}^t\Bigg]}_{\text{(iii)}} \qquad (3.1)$$

in which the expected revenue ((i)+(ii)) is discounted by the agent's discount rate of capital, β_h.

The model of the agent's behavior that maximizes profits over the long term is proposed as the following problem of deterministic optimization over the present value of profit:

$$\max_{G_{hi}} \sum_{t=0}^{\infty} \beta_h^{(t)} \Pi_h^t(\beta_h, G_i^t, q^t, \bar{r}^t, p^t, p^{t+1}) \qquad (3.2)$$

s.t.

$$K_h^t(p^t) - \sum_i G_{hi}^t = 0 \qquad \forall t \qquad (\theta^t) \qquad (3.2.1)$$

$$G_{hi}^t \geq 0 \qquad \forall t, \forall i \qquad (\delta_i^t) \qquad (3.2.2)$$

Where θ^t and δ_i^t are the Lagrange multipliers, and K_h^t is the agent's capital at the beginning of each period.

Assuming that at the beginning of each period agents liquidate their stock at equilibrium prices, the capital of an agent *h* in period *t*, denoted by $K_h^t \in \mathbb{R}^+$, is defined as the dynamic function:

$$K_h^t(p^t) = (1 + q^{t-1})G_{h0}^{t-1} + \sum_{i \neq 0} \left(\frac{\bar{r}_i^{t-1} + p_i^t}{p_i^{t-1}} \right) G_{hi}^{t-1} + \hat{y}_h^{t-1} \qquad (3.3)$$

The profit of equation (3.1) is optimized by making optimal decisions according to problem (3.2) subject the capital constraint given by equation (3.3). This relationship represents the market dynamics, where in a given period *t*, the capital K_h^t constrains the total expenditure in that period, and the profit made by decisions the same period *t* will affect the agent's capital for the next one (*t*+1).

Notice that maximizing the expected value of profits in the long run subject to conditions (3.2.1) and (3.2.2) is equivalent to maximize the present value of the capital in the long term period subject to the same conditions, considering only actions the agent makes in the real estate market (i.e. exclude the agents available income assumed independent of his choices in the real estate market).

3.2. Optimality conditions.

Given the inter-temporal dependency in the agents' decisions ($K^t(G^{t-1})$), it is necessary to take into account how decisions taken on a given period are going to affect the future. Problem (3.2) can be reformulated assimilating the agents decisions to the dynamic problem of invest and save proposed by Stockey et. al. (1989), which would write (3.2) in recursive terms upon defining a function of value V as follows:

$$V(\xi^t) = \max_{G^t} \left[\beta \left[(1 + q^t)G_0^t + \sum_{i \neq 0} \left(\frac{r_i^t + p_i^{t+1}}{p_i^t} \right) G_i^t \right] - \sum_i G_i^t + \beta V(\xi^{t+1}) \right]$$

s.t.

$$K^t - \sum_i G_i^t = 0 \qquad \forall t \qquad (\theta^t) \qquad (3.4)$$

$$G_i^t \geq 0 \qquad \forall t, \forall i \qquad (\sigma_i^t)$$

where ξ^t represents the set of variables (K^t, G_i^t) and the sub index *h* is omitted in order to simplify notation. This dynamic function represents the profit maximization of an agent on a given period, plus the expected value of its investments in future periods (discounted by β).

In order to get the agent's optimum expenditure on each period, the first order conditions of problem (3.4) are obtained:

$$\beta(1+q^t)-1+\beta\frac{\partial V(\xi^{t+1})}{\partial G_0^t}-\theta^t+\sigma_0^t=0 \qquad \text{if } i=0,\forall t \tag{3.5}$$

$$\left(\frac{r_i^t+p_i^{t+1}}{p_i^t}\right)-1+\beta\frac{\partial V(\xi^{t+1})}{\partial G_i^t}-\theta^t+\sigma_i^t=0 \qquad \forall i\neq 0,\forall t \tag{3.6}$$

Furthermore, it is also necessary to consider the effects that decision made in period *t* will have on the future. For this reason, it is useful to consider the envelope theorem to quantify these effects, as shown below:

$$\frac{\partial V(\xi^{t+1})}{\partial G_0^t}=(1+q^t)\theta^{t+1} \qquad \text{if } i=0,\forall t \tag{3.7}$$

$$\frac{\partial V(\xi^{t+1})}{\partial G_i^t}=\left(\frac{r_i^t+p_i^{t+1}}{p_i^t}\right)\theta^{t+1} \qquad \forall i\neq 0,\forall t \tag{3.8}$$

By replacing (3.7) and (3.8) into (3.5) and (3.6) respectively, the following Euler equations for the agent's problem are obtained:

$$\beta(1+q^t)-1+\beta(1+q^t)\theta^{t+1}-\theta^t+\sigma_0^t=0 \qquad \text{if } i=0,\forall t \tag{3.9}$$

$$\left(\frac{r_i^t+p_i^{t+1}}{p_i^t}\right)-1+\beta\left(\frac{r_i^t+p_i^{t+1}}{p_i^t}\right)\theta^{t+1}-\theta^t+\sigma_i^t=0 \qquad \forall i\neq 0,\forall t \tag{3.10}$$

Given that θ^{t+1} is unique among all options, from (3.9) and (3.10) it is possible to obtain the Lagrange multipliers equations that rule the agent's optimal behavior:

$$\sigma_0^t(\sigma_i^t)=\left(\frac{(1+q^t)p_i^t}{r_i^t+p_i^{t+1}}\right)\sigma_i^t+(1+\theta^t)\left(1-\frac{(1+q^t)p_i^t}{r_i^t+p_i^{t+1}}\right) \qquad \forall i,\forall t \tag{3.11}$$

$$\sigma_i^t(\sigma_0^t)=\left(\frac{r_i^t+p_i^{t+1}}{(1+q^t)p_i^t}\right)\sigma_0^t+(1+\theta^t)\left(1-\frac{r_i^t+p_i^{t+1}}{(1+q^t)p_i^t}\right) \qquad \forall i,\forall t \tag{3.12}$$

$$\sigma_i^t(\sigma_j^t)=\left(\frac{(r_i^t+p_i^{t+1})p_j^t}{(r_j^t+p_j^{t+1})p_i^t}\right)\sigma_j^t+(1+\theta^t)\left(1-\frac{(r_i^t+p_i^{t+1})p_j^t}{(r_j^t+p_j^{t+1})p_i^t}\right) \qquad \forall i,\forall j,\forall t \tag{3.13}$$

Equations (3.11), (3.12) and (3.13) impose the first order and Euler conditions simultaneously for the optimum. From these equations, the following definitions and the complementary slackness conditions of problem (3.4) yield Lemma 1.

Using equation (3.1) we define R_i^t as the expected unitary returns by option type, given by:

$$R_i^t = \begin{cases} \beta(1+q^t) - 1 & \text{if } i = 0 \\ \beta\left(\dfrac{r_i^t + p_i^{t+1}}{p_i^t}\right) - 1 & \sim \end{cases} \tag{3.16}$$

from where it is clearly seen that $\Pi_h^t = \sum_i R_i^t G_i^t$.

And from (3.16) we define Ω_*^t as the set of highest unitary returns, given by:

$$\Omega_*^t = \underset{i \in I}{\operatorname{argmax}}\{R_i^t\} \tag{3.17}$$

<u>Lemma 1:</u>

$$G^t \text{ is the optimal solution of (3.2)} \iff G^t \text{ is feasible such as } \begin{cases} G_i^t \neq 0 & \text{if } i \in \underset{j \in I}{\operatorname{argmax}}\{R_j^t\} \\ G_i^t = 0 & \text{if } i \notin \underset{j \in I}{\operatorname{argmax}}\{R_j^t\} \end{cases}$$

<u>Proof:</u>

Given a feasible expenditure vector G^t such as $\begin{cases} G_i^t \neq 0 & \text{if } i \in \operatorname{argmax}_{j \in I}\{R_j^t\} \\ G_i^t = 0 & \text{if } i \notin \operatorname{argmax}_{j \in I}\{R_j^t\} \end{cases}$

$\iff \begin{cases} \sigma_i^t = 0 & \text{si } i \in \operatorname{argmax}_{j \in I}\{R_j^t\} \\ \sigma_i^t \neq 0 & \text{si } i \notin \operatorname{argmax}_{j \in I}\{R_j^t\} \end{cases}$ (complementary slackness conditions of problem (3.4))

$\iff \sigma_j^t(\sigma_i^t) > 0 \quad \forall i \in \operatorname{argmax}_j\{R_j^t\}, \forall j \notin \operatorname{argmax}_j\{R_j^t\}$ (conditions (3.11), (3.12) and (3.13))

$\iff G^t$ is the optimal solution of problem (3.2). ■

Lemma 1 states that, given the linear objective function of problem (3.4) in the decision variable $G_i^t \ \forall i$, then the solution will always be extreme. Therefore, the agent will allocate its whole capital to the highest unitary return option(s) at each period. If there are two or more options with the same unitary return, then these options are indifferent and the agents' optimum has several solutions.

Therefore, we conclude that the optimal expenditure solution vector G^t, conditional on the given prices set is:

(a) If $\operatorname{card}(\Omega_*^t) = 1 \implies$ The optimal solution to problem (3.2) is unique and it is given by:

$$G^t(p^t) = \begin{cases} G_i^t = K^t(p^t) & \text{if } i \in \Omega_*^t \\ G_i^t = 0 & \text{if } i \notin \Omega_*^t \end{cases} \tag{3.18}$$

(b) If $\mathrm{card}(\Omega_*^t) > 1 \implies$ There are infinite optimal solutions to problem (3.2) which arise from the maximum total expenditure condition (3.2.1) and are given by:

$$G^t(p^t) = \begin{cases} \sum_i G_i^t = K^t(p^t) & \text{si } i \in \Omega_*^t \\ G_i^t = 0 & \text{si } i \notin \Omega_*^t \end{cases} \tag{3.19}$$

Finally, given that problem (3.2) is solved by choosing the best unitary return(s) option(s), its solution is equivalent to the discrete problem of choosing the best unitary returns among all options in each period, given by:

$$\max_{i \in I}\left(R_{hi}^t\right) \tag{3.20}$$

which is the rule that agents can apply ensuring him/her to attain maximum profit.

3.3. The stochastic model.

In order to model idiosyncratic heterogeneity among agents' behavior, the fact that agents do not have complete information about the market, and the myopic assumption which says that agents do not know future rents and future prices (r_{hi}^t and p_{hi}^{t+1} respectively), it is assumed that expected unitary returns are random variables of type $\tilde{R}_{hi}^t = R_{hi}^t + \varepsilon_{hi}$, where the random term ε_{hi} is modeled as an IID Gumbell distribution that arise from the auction process from where rents are obtained that take place in the land use market, due to both misinformation and coordination problems.

The solution to problem (3.20) with random returns is expressed by the probability that agent *h* will maximize choose option *i,* in period *t*, as in the following multinomial logit expression:

$$\mathcal{P}_{i|h}^t = \frac{\exp\left(\mu R_{hi}^t\right)}{\sum_j \exp\left(\mu R_{hj}^t\right)} \tag{3.21}$$

Therefore, the solution for each agents' optimal expenditure in each period for each option *i*, is given by:

$$G_{hi}^t = K_h^t \frac{\exp\left(\mu R_{hi}^t\right)}{\sum_{j \in I} \exp\left(\mu R_{hi}^t\right)} \qquad \forall h, i, t \tag{3.22}$$

3.3. Equilibrium.

The notion of equilibrium used in the model can be explained as "the total number of stock among the agents must be equal to the total exogenous supply in each period". This condition is written as the following:

$$\sum_h N_h^t S_{hi}^t(r^t, p^t) = \bar{O}_i^t(p_i^t) \quad \forall i, \forall t \qquad (3.23)$$

where $\bar{O}_i^t(p_i^t)$ is a subset of the total supply available in the market at period t defined by $\bar{O}_i^t(p_i^t) = O_i^t \rho_i^t(p_i^t)$. In this equilibrium condition the two related submarkets affect the solution endogenously: the stock production submarket defines the total supply as an elastic function of selling prices, with ρ_i^t the proportion offered in the market at prices p_i^t, with $\frac{\partial \rho_i^t}{\partial p_i^t} > 0 \ \ \forall i, t$ and $\rho_i^t \in [0,1] \ \ \forall i, t$. On the other hand, the land use submarket defines rents, which in turn defines the optimal stock for each agent.

Given the existent relationship between agent's stock and agent's expenditure presented in equation (2.2), and given condition (3.23), the fixed point set of equations in the price vector that solve the problem are finally obtained as:

$$p_i^t(p^t, p_h^{t+1}, r_h^t, O_i^t) = \frac{1}{O_i^t \rho_i^t(p_i^t)} \sum_h N_h^t K_h^t \mathcal{P}_{i|h}^t \quad \forall i, \forall t \qquad (3.24)$$

We can analyze the sensitivity of the selling price function shown in (3.24) to market changes:

(i) If submarket *i* gets thicker, then:

$$\frac{\partial p_i^t}{\partial O_i^t} = -\frac{1}{{O_i^t}^2 \rho_i^t(p_i^t)} \sum_h N_h^t K_h^t \mathcal{P}_{i|h}^t \ < \ 0 \ \ \forall i, t \qquad (3.25)$$

and prices decrease as expected.

(ii) If demand from investors *h* gets thicker, then:

$$\frac{\partial p_i^t}{\partial N_h^t} = \frac{K_h^t \mathcal{P}_{i|h}^t}{O_i^t \rho_i^t(p_i^t)} \ > \ 0 \ \ \forall i, t \qquad (3.25)$$

and prices increase, as expected, proportionally to the demand in each submarket.

(iii) If total wealth increases in proportion α, then

$$\Delta p_i^t = \alpha p_i^t \ > 0 \ \ \forall i, t \qquad (3.26)$$

which shows how real estate prices capitalize wealth into prices.

(iv) If a single agent h increases its wealth,

$$\frac{\partial p_i^t}{\partial K_h^t} = \frac{N_h^t \mathcal{P}_{i|h}^t}{O_i^t \rho_i^t (p_i^t)} \qquad > \; 0 \;\; \forall i, t \tag{3.27}$$

which demonstrates that the price increment on a given option depends proportionally on the demand of that agent for that option.

3.5. Existence and uniqueness of equilibrium.

In this section, to simplify the presentation we consider the case where prices are high enough to make developers to offers all the available stock in the market ($\rho_i^t = 1 \;\; \forall i, t$), and that there is only one agent per cluster ($N_h^t = 1 \;\; \forall i, t$)

In order to analyze the existence, unity and convergence of the fixed point set of equations in (3.24), we define the function F_i^t for each option *i*, in each period *t* as:

$$F_i^t(p^t) = p_i^t - \frac{1}{O_i^t} \sum_h \left[\mathcal{P}_{i|h}^t \left(\gamma_{hi} + \sum_{i \neq 0} \frac{p_i^t}{p_i^{t-1}} G_{hi}^{t-1} \right) \right] \tag{3.28}$$

where:

$$\gamma_{hi} = (1 + q^{t-1}) G_{h0}^{t-1} + \sum_{i \neq 0} \frac{r_i^{t-1}}{p_i^{t-1}} G_{hi}^{t-1} + \hat{y}_h^{t-1} \tag{3.29}$$

$$\mathcal{P}_{i|h}^t(p^t) = \frac{\exp(\mu R_{hi}^t)}{\sum_j \exp(\mu R_{hj}^t)} \tag{3.30}$$

$$R_{hi}^t = \beta_h \left(\frac{r_i^t + p_i^{t+1}}{p_i^t} \right) - 1 \tag{3.31}$$

3.5.1. Existence.

Here we analyze the co-domain of function $F_i^t(p^t)$:

(i) $p_i^t \to 0^+ \Longrightarrow R_{hi}^t \to +\infty \Longrightarrow \mathcal{P}_{i|h}^t \to 1$

$$\Longrightarrow \lim_{p_i^t \to 0^+} F_i^t(p^t) = -\frac{1}{O_i^t} \sum_h \gamma_{hi} \qquad < 0 \tag{3.32}$$

(ii) $p_i^t \to +\infty \Longrightarrow R_{hi}^t \to 0^+ \Longrightarrow \mathcal{P}_{i|h}^t \to 0^+$

$$\Rightarrow \lim_{p_i^t \to 0^+} F_i^t(p^t) = +\infty \qquad > 0 \tag{3.33}$$

Thus, the image of function $F_i^t(p^t)$ is:

$$\mathrm{Im}\left(F_i^t(p^t)\right) = \left[-\frac{1}{O_i^t}\sum_h \gamma_{hi}, +\infty\right] \quad \forall i, \forall t \tag{3.34}$$

Given that $F_i^t(p^t)$ is continuous in all its domain $\forall i, t$, the existence of a solution to the equilibrium problem (3.24) is ensured.

3.5.2. Uniqueness.

We need to ensure that function $F_i^t(p^t)$ is monotone, so we study the critical conditions that make its gradient equal to zero. These conditions are:

$$\frac{\partial F_i^t(p^t)}{\partial p_i^t} = 0$$

$$\Leftrightarrow \mu_i^{t*}(p^t) = \frac{O_i^t - \sum_h \mathcal{P}_{i|h}^t\left(\frac{G_{hi}^{t-1}}{p_i^{t-1}}\right)}{\sum_h \mathcal{P}_{i|h}^t\left(1-\mathcal{P}_{j|h}^t\right)\beta_h\left[\frac{\frac{\partial p_i^{t+1}}{\partial p_i^t}p_i^t - \left(r_i^t + p_i^{t+1}\right)}{\left(p_i^t\right)^2}\right]\left(\gamma_{hi} + \sum_{i\neq 0} p_i^t\left(\frac{G_{hi}^{t-1}}{p_i^{t-1}}\right)\right)} \quad \forall i, \forall t \tag{3.35}$$

Therefore, from equation (3.32) the monotonicity conditions are obtained:

$$\text{(i)} \quad \mu \le \mu_i^{t*}(p^t) \quad \forall i, \forall t \Rightarrow \frac{\partial F_i^t(p^t)}{\partial p_i^t} \ge 0$$

$$\Rightarrow F_i^t(p^t) \text{ increases monotonically in its domain } \forall i, \forall t. \tag{3.36}$$

$$\text{(ii)} \quad \mu \ge \mu_i^{t*}(p^t) \quad \forall i, \forall t \Rightarrow \frac{\partial F_i^t(p^t)}{\partial p_i^t} \le 0$$

$$\Rightarrow F_i^t(p^t) \text{ decreases monotonically in its domain } \forall i, \forall t. \tag{3.37}$$

From condition (3.34), if condition (3.36) is met, then it is possible to ensure the existence of a unique solution for the set of equations presented in (3.24) that solve the equilibrium problem. Given that μ represents the inverse of the dispersion of the distribution of the random variable ε_{hi}, strict positivity must be ensured. Using equation (3.35), strict positivity condition is met only if:

$$\mu_i^{t*}(p^t) > 0 \quad \forall i, \forall t$$

$$\Leftrightarrow \left(\left[O_i^t > \sum_h \mathcal{P}_{i|h}^t \left(\frac{G_{hi}^{t-1}}{p_i^{t-1}} \right) \right] \wedge \left[\frac{\partial p_i^{t+1}}{\partial p_i^t} p_i^t > \left(r_i^t + p_i^{t+1} \right) \right] \right) \vee \left(\left[O_i^t < \sum_h \mathcal{P}_{i|h}^t \left(\frac{G_{hi}^{t-1}}{p_i^{t-1}} \right) \right] \wedge \left[\frac{\partial p_i^{t+1}}{\partial p_i^t} p_i^t < \left(r_i^t + p_i^{t+1} \right) \right] \right) \quad \forall i, \forall t \qquad (3.38)$$

Therefore, existence of a unique solution for the equilibrium problem is ensured if parameter μ is strictly positive and small enough. This condition can be written as:

$$\text{The set of equations (3.24) have a unique solution } \forall i \quad \Leftrightarrow \quad \text{the parameter } \mu \in]0, \bar{\mu}[\qquad (3.39)$$

where $\bar{\mu} = \min\{\mu_i^{t*}\}$ with μ_i^{t*} given by (3.35).

4. SIMULATIONS

4.1. Myopic assumption.

We assume that agents estimate the future market prices by considering the historic information of the real estate market, which is known as the myopic assumption. Therefore, the following rules were considered for the agents' estimations of future net rents ($\bar{r}_{hi}^t$) and future prices (p_{hi}^{t+1}) respectively:

$$\bar{r}_{hi}^t \left(\lambda_h, \bar{r}_i^{t-1}(r_i^{t-1}), \bar{r}_i^{t-2}(r_i^{t-2}) \right) = \lambda_h \bar{r}_i^{t-1} + (1 - \lambda_h) \bar{r}_i^{t-2} \quad \forall i, \forall t \qquad (4.1)$$

$$p_{hi}^{t+1}(\lambda_h, p_i^t, p_i^{t-1}) = p_i^t + \lambda_h (p_i^t - p_i^{t-1}) \qquad \forall i, \forall t \qquad (4.2)$$

where $\lambda_h \in [0,1]$ and it represents the risk agents take when making their estimates. In this way, if $\lambda_h \approx 0$ represents a risk adverse agent; if $\lambda_h \approx 1$ represents risk prone agent.

Under these assumptions, the agents' expected unitary returns for each period will be:

$$R_{hi}^t = \begin{cases} \beta_h (1 + q^t) - 1 & \text{si } i = 0 \\ \beta_h \left(\frac{\bar{r}_{hi}^t + p_{hi}^{t+1}}{p_i^t} \right) - 1 & \sim \end{cases} \qquad (4.3)$$

4.2. Parameters.

The following tables present the required input parameters that were used in the three simulations presented here. It is assumed that there is only one agent per cluster ($N_h^t = 1 \quad \forall i, t$). Table I presents the general parameters that define the competition level, market size, time horizon, dispersion and equilibrium level respectively.

Table I – General parameters.

Total number of agents	H	3
Total number of options	I	4
Total number of periods	T	40
Distribution's scale parameter	μ	1.85
Minimum tolerance in order to reach equilibrium	mintol	0.5
Maximum number of iterations for equilibrium	maxiter	1000

Table II presents the parameters that define agents, ordered by decreasing wealth (disposable income plus initial capital) and capital discount rate, from h = 1 to h = 3. It is also considered that agent h = 3 will be the most risk prone.

Table II – Agents' parameters.

		Agent [h]		
		h = 1	h = 2	h = 3
Capital discount rate	β_h	0.90	0.85	0.80
Price estimation parameter	λ_h	0.50	0.20	0.85
Disposable income	$\hat{y}_h$	0.50	0.25	0.10
Initial capital	K_h^0	5.00	2.50	1.00

4.3. Simulations.

In order to generate comparable simulations, we defined a baseline (scenario 1) that was then subject to shocks on supply (scenario 2) and on agents' capital (scenario 3).

The baseline scenario is defined by a bank's interest rate $q^t = \begin{cases} 0.1 & \text{if } t \in [1,10] \\ 0.00001 & \text{if } t \in [11,29] \\ 0.01 & \sim \end{cases}$; exogenous rents $r_1^t = 0.05 \ \forall t$, $r_2^t = \begin{cases} 0.1 & \text{if } t \in [10,20] \\ 0.0 & \sim \end{cases}$, $r_3^t = \begin{cases} 0.02 & \text{si } t \in [1,15] \\ 0.0 & \sim \end{cases}$, $r_4^t = \begin{cases} 0.25 & \text{si } t \in [25,40] \\ 0.0 & \sim \end{cases}$; and a constant exogenous supply $O_i^t = 10 \ \ \forall i, \forall t$. Consumers capitalize over time (Figure 1.2) starting from different initial wealth (*h* = 1 the richer and *h* = *3* the poorer) at rates according to their exogenous income. As expected equilibrium prices (Figure 1.2) increase over time capitalizing on incremented wealth but subject to sudden changes that follow rent shocks. Profits shown in Figure 1.3, once they stabilize remain positive, but affected by shocks and are differentiated by the agents' wealth. Such differentiation is also reflected in market shares, which remain very stable along time.

Scenario 2 assumes a supply variations defined by the exogenous shocks: $O_1^t = 10 \ \forall t$, $O_2^t = 10 \ \forall t$, $O_3^t = \begin{cases} 8 & \text{si } t \in [12,22] \\ 9 & \text{si } t \in [23,32] \\ 10 & \sim \end{cases}$, and $O_4^t = \begin{cases} 10 & \text{si } t \in [1,29] \\ 15 & \sim \end{cases}$. Following shown in equation (3.25), on period *t* = 12 we observe a rise in the price of option *i* = 3 (Figure 2.2) due to the disappearance of that option's supply. At period *t* = 30 observe a drop in the price of option's *i* = 4 due to the increase in supply. Profits also drops at this point in time drastically for the wealthier and market dominant group *h* = 1.

Scenario 3 assumes that on period t = 20 the agent h = 1 suddenly looses almost all its capital. The same shock is assumed on period t = 30, where agents h = 1 and h = 2 loose almost all their capital; these shocks are depicted in Figure 3.1. As expected from equations (3.26) and (3.27), in Figure 3.2 we observe general drops in prices and profits following these shocks on periods t = 20 and t = 30. Notably, market shares and wealth change hands (Figure 3.3), first from h = 1 to h = 2 (and less so to h = 3) but then agent h = 3 dominates the market.

Figures of simulation results.

Base Scenario: Shocks on Rents and Bank's Interest Rate	**Base Scenario + Shocks on the Supply**	**Base Scenario + Shocks on Agents' Capital**

Figure 1.1

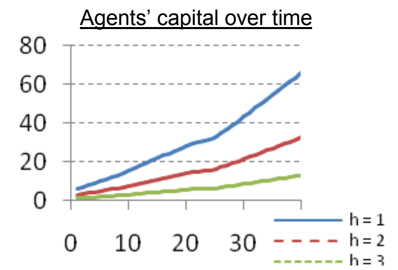


Figure 2.1

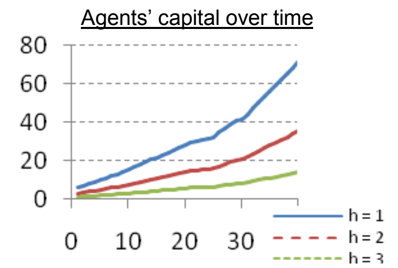


Figure 3.1

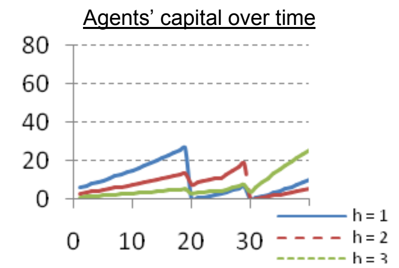


Figure 1.2

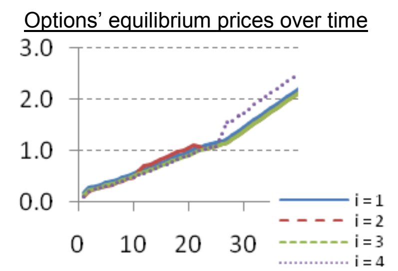


Figure 2.2

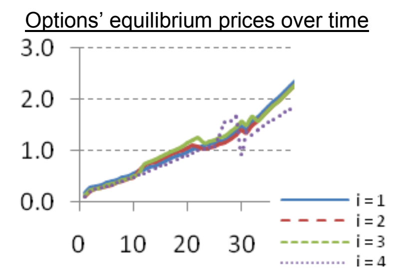


Figure 3.2

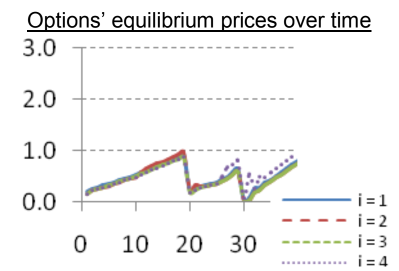


Figure 1.3

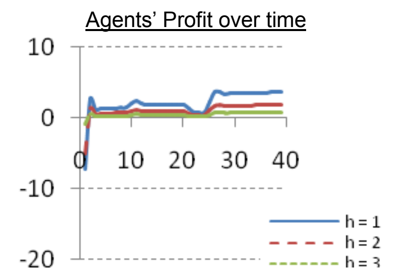


Figure 2.3

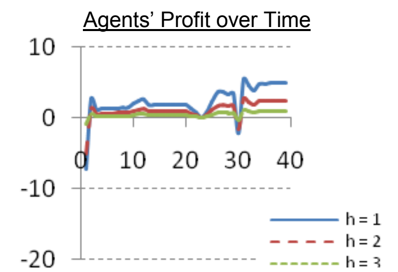


Figure 3.3

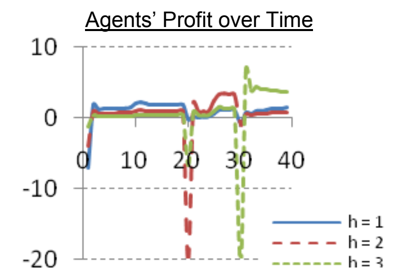


Figure 1.4

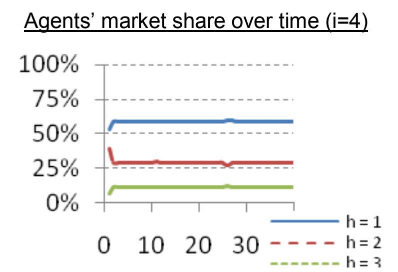


Figure 2.4

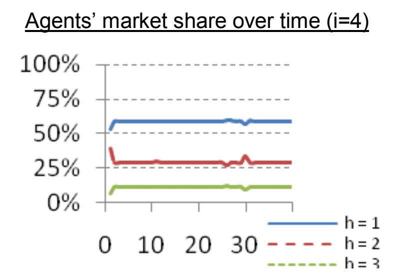
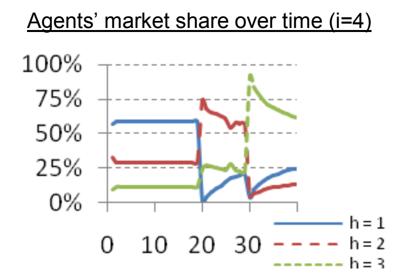


Figure 3.4



5. CONCLUSIONS

We have formulated and analyzed an economic equilibrium model of the real estate ownership market. The approach is consistent with the related submarkets: stock production and land use markets. It considers the system dynamics by modeling the inter-temporal decision of investors of maximizing profit in the long term. It also includes the financial market as fixed unitary return option (the *bank*), or the savings option, thus defining a closed market.

The model is able to forecast the agents' reaction to market stimuli, such as prices and interest rates, and it is sensitive to the level of risk they take with regards to the lack of information of the future performance of the market. The model also forecasts bankrupts of investors, identifying their category, and the takeover of some investors of some specific sub-markets. The price equilibrium is also subject to stimuli from related markets, including the thickness of the market on supply and demand, interest rates and other shocks in the capital market.

Regarding the interaction with the land use market, the model provides the expected distribution of agent in the real estate market, which is the required information to define the partition of the residents that are renters of their homes. This distribution changes along time according to demographic and economic conditions.

The assumption that investor liquidate all their stock at the beginning of each period needs to be adapted to incorporate realistic conditions in the loan market that prevent investors to sell during a period. This can be easily introduced by a loan cost that decreases on time after the investment.

Even though this model was designed and presented for the real estate ownership submarket, it can be easily adapted to a finance model where agents decide their optimal capital's distribution among a group of discrete investment options. An interesting feature of the model, which represents a progress in finance models, is the incorporation of market clearing through equilibrium prices, which define the unitary returns of the options perceived by the agents.

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