A DESIGN MODEL FOR THE BUS BRIDGING PROBLEM

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Abstract During daily operations in a dense rapid transit network passengers may deviate from their usual routes because of incidents leaving one or more transit lines out of service. A common practice in order to alleviate these disruptions is by setting bus-bridging services amongst affected stations. Because these kind of services must operate at high levels of demand, design models of these complementary lines must take into account service under congested operation conditions. In this paper a model in order to determine assignment of units to the lines of the bus-bridging system is developed under a linear integer programming formulation. The model minimizes total setting and exploitation costs of the bus-bridging service plus social costs by means of an estimation of the user's travel time. The model assumes a system optimum passenger's flow distribution and by means of suitable approximations based on queueing theory it takes into account waiting times of passengers at stations, prevention of blocking at stations and queueing of buses and also a stations with a limited capacity for allocating waiting queues of passengers.

Keywords: Bus operations, transit modelling, integer linear programming

1 Introduction

Disruption of services in rapid transit systems or even metro systems in large urban areas may affect a considerable part of the travel demand, which specially in cases of home-to-work trips and other mandatory purposes, must take place on the transport network. Anyway many of this demand will switch from transportation mode directly from the origin increasing car usage and road transportation and the amount of this travel demand will be at its maximum possible level if there is not any mean of using the rapid transit system as a reasonable way to reach its destination. A recourse for alleviating this situation is to bridge terminal stations of disrupted sections of the regular service (either rapid transit or metro) by means of auxiliary bus services. Because the alighting capacity of buses will always be below possibilities of capturing the entire level of disrupted demand it must be expected that these bridging services will operate at a high degree of congestion. Additionally, if the disrupted portion of the regular transportation service is a grid subnetwork, then the need of an accurate dimensioning of the bridging services is not a straightforward problem at all. Dimensioning of services for bus lines has always been tackled in the scientific literature for the case of regular services in conditions where congestion of the bus network is at a moderate level. Only recently in [9] the problem of bridging services has been directly taken into consideration, although the paper is more addressed to the design of a decision support system for the bridging service rather than an accurate analysis of the performance of the system and a modeling of the congestion.

In this paper a model for dimensioning a predetermined set of bus lines bridging disrupted stations is presented in the last section and examples of application to realistic networks will be presented in the conference. The model is based on a formulation in integer linear programming. In section 7 the representation of the bridging network is described. Section 3 provides a generic enumeration of the distinct steps and phases that can be identified in the operation of bus systems times which will be used as reference in further sections of the paper. Sections 4 and 5 describe in detail how different congestion factors are modeled by means of suitable approximations based on queueing theory. In particular, it takes into account waiting times of passengers at stations, prevention of blocking at stations and queueing of buses and also a stations with a limited capacity for allocating waiting queues of passengers. The model minimizes total setting/operational costs of the bus-bridging service plus social costs by means of an estimation of the user's travel time. The model assumes a system optimum passenger's flow distribution so solutions result from a balancing between social costs and setting/operational costs.

2 Network flows model

Passenger flows go through a directed graph $G=(N,A)$, whose structure is sketched in figure 1. The set of nodes N splits into two subsets, N^G and $N \setminus N_G$. Nodes in the set N^G and links $a=(i,j)$ so that $i \in N_G$, $j \in N_G$ are used to model transfer movements or simply trips balking from the bus transportation system and carried out using an alternative transportation mode.

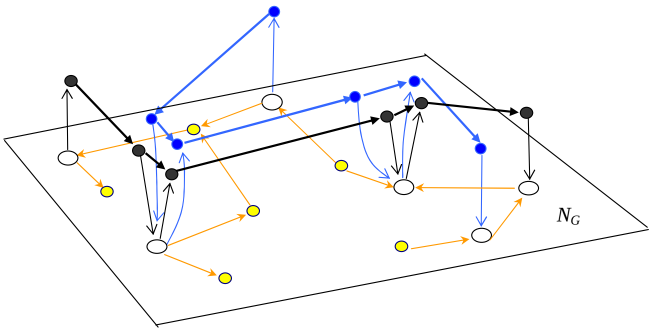

Figure 1. Network representation by means of the graph $G=(N,A)$.

On the graph, incoming and outgoing nodes from a given one $i \in N$ are designated by $E(i), I(i)$ respectively. Bus stops or stations are represented by single nodes making up a subset $\hat{N}_G \subseteq N_G$. By L it is designated the set of bus lines and $\Pi_\ell = \{b_1,\ldots,b_{n_\ell}\}$ is the ordered set of n_ℓ bus stops or stations $b_j \in \hat{N}_G$ on line $\ell \in L$. L_b is the set of bus lines containing stop or station b. The subset of links A_b is made up by links $a=(i,j)$ so that, neither $i \in N_G$ nor $j \in N_G$ and a is an entering link to station b:

$$A_b = \{\, (i,j) \in A \,|\, i,j \notin N_G, \exists (j,k) \in E(j)\; t.q.\; k \in I(b) \,\} \tag{1}$$

A_ℓ is the set of links modeling line $\ell \in L$ so that, $i,j \notin N_G$

$$A_\ell = \{\, (i,j) \in A \,|\, i,j \notin N_G, (i,j)\ \textit{belong to line}\ \ell \} \tag{2}$$

By A_G it is designated the set of links whit nodes $i,j \in N_G$.

$$A_G = \{ (i, j) \in A | i \in N_G, j \in N_G \} \tag{3}$$

Figure 2 shows in more detail the set of links modeling alighting and boarding operations at stations. Links $a = (b, j)$, $b \in \Pi_\ell$, $j \notin N_G$ capture boarding and waiting time at a station for passengers willing to board on servers of line ℓ For a given station b belonging to line ℓ, boarding link from $b \in N_G$ to line ℓ will be designated by $a(\ell, b)$. For a boarding link a , link $x(a)$ denotes the link on which passengers wait on board of the server without alighting at that station, whereas by $y(a)$ it is denoted the corresponding alighting link.

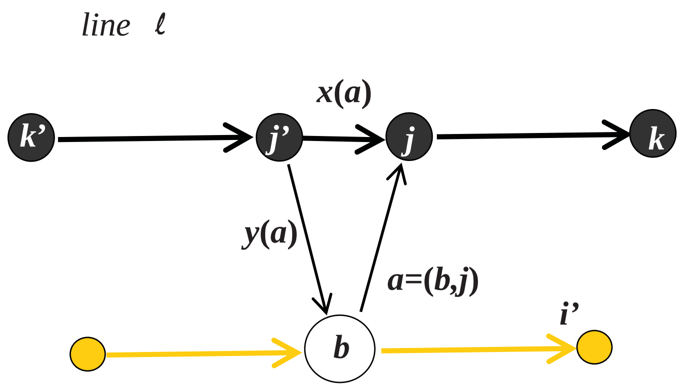

Figure 2. Detail of the expanded graph for modeling bus lines and stations. $b, i' \in N_G$. $(k', j') \in A_b$. $x(a) = (j', j), (j, k) \in A_\ell$.

The set of lines incoming to a station b can be defined as:

$$L_b = \{\ell \in L | A_\ell \cap A_b \neq \varnothing\} \tag{4}$$

The set of O-D pairs W is defined form nodes in N_G, which are not necessarily stations. $\omega = (i, d) \in W$, $i \in N_G$, $d \in N_G$. D is the set of nodes which are destination. The O-D trip matrix will be designated by $g_\omega, \omega \in W$. Indexes ω, or explicitly (i, d), will be used for an O-D pair when considered convenient.

Flows will be organized in commodities, one per each destination, so that flow on a link $a \in A$ for a destination $q \in D$ will be designated by v_a^d . Balance equations for flows per destination $d \in D$ at a node $i \in N$ will be:

$$\sum_{j \in E(i)} v_{ij}^d - \sum_{k \in I(i)} v_{ki}^d = \begin{cases} -\sum_{(i,d) \in W} g_{id} & if \quad i = d, \quad \forall i \in N, d \in D \\ g_{id}, & if \quad i \neq d, (i, d) \in W \\ 0 & if \quad (i, d) \notin W \end{cases} \tag{5}$$

By adding non-negativity conditions $v_a^d \geq 0$ for flows on links to previous relationships ((5)), the following polyhedra are defined:

$$V^d = \left\{ v^d \in R_+^{|A|} | \; v^d = (\ldots, v_{i,j}^d, \ldots; (i, j) \in A), v_{i,j}^d \; verify \; (5) \right\}, d \in D \tag{6}$$

3 Operational times outline

The model described in this paper comprises most of the different operational times in bus systems. Because congestion has a fundamental role and because realistic solutions are pursued. These operational times are considered at different levels of accuracy. In [10] and in [2] a good description of them is given.

Taking as a base reference [10] the following operational times for buses and passengers are considered:

Operational times V for vehicles

• Station to station time (including breaking when approaching a station and acceleration for exiting the station)

• Maneouver time entering the station

• Dwell time in the station:

- Waiting time queueing for a berth

- Maneouver time for entering/exiting from the berth

- PST Passenger Service Time (PST). Boarding and alighting of passengers (includes opening/closing door times)

• Blocking time. After PST it may happen that the server cannot leave the berth because of queueing for exiting the station.

• Maneouver time for exiting the station.

• layover time at the end of the line (usually at one or two terminal stations)

Passenger's time P

Three cases should be taken into account:

• Passengers waiting on board. Their times are those of the vehicle they are on board. V1 + V2 + V3 + V4 + V5.

• Passengers alighting from the bus. These passengers experience times V1, V2, V3(b) (parking-only access) and V3(a); in addition the individual alighting time must be taken into account. This time can be modeled a) proportional to the amount of passengers alighting b) constant. Option a) is more realistic but in order to keep the formulation of the design model within the linear-integer programming framework this approach makes the resulting formulation even more complex. Anyway, alighting time usually is much smaller than the remaining operational times and it is also smaller that boarding times.

• Boarding of passengers. They experience:

- Waiting time for a server to arrive at station. This originates a passengers queue for each of the lines stopping at the station (strategies in the sense of Florian and Spiess [18] are not considered and each of the passengers has in mind only one line to board on)

- Boarding time. This time can be modeled simply as $p+qv$, where v is the number of passengers boarding on a server and p and q are constants. This time overlaps with time referred to in V3(c). Usually boarding time is greater that alighting time. One of the earliest references for modeling PST is that of [12].

- times V3(b), V3(d) and V4
- time V1

4 Congestion aspects taken into account in the model

4.1 Grouping buses

As the model described in this paper is intended to cover disruption of regular services it will be assumed that available units may come from different sources so that a degree of heterogeneity in the fleet composition might be assumed. The model considers that previously to the assignment of buses to the lines, buses can be grouped forming, say, packets so that relevant characteristics such as total capacity of buses in a packet is made as homogeneous as possible across the total population of packets.

Grouping buses into packets may be addressed by means of solving a typical clustering problem. Assume that K buses are available, each of them with a capacity $c_i, i = 1,2,...K$, (c_i = total packet capacity in passengers). Assume that a desirable packet capacity $\bar{c}$ is to be achieved with p packets (clearly if $C = \sum_{j=1}^{n} c_j$ is the total fleet capacity, then there must hold $C = p\bar{c}$). If now x_{ij} is a binary variable with value 1 if bus i is assigned to packet j and 0 otherwise, the following mathematical program sets p packets so that individual packet capacity is of minimal variance

$$
\begin{aligned}
Min_x \quad & \sum_{j=1} p(\sum_{i=1}^{n} x_{ij} c_i - \bar{c})^2 \\
s.t. \quad & \sum_{i=1}^{n} x_{ij} \geq 1, \forall j \\
& \sum_{j=1} p x_{ij} = 1, \forall i \\
& \sum_{j=1} p \sum_{i=1}^{n} x_{ij} c_i = p\bar{c} \\
& x_{ij} \in \{0,1\}
\end{aligned}
\tag{7}
$$

In this paper it will be assumed that the population of packets that have been formed cannot be subdivided in clusters each of them having average characteristics significatively different from each other. We note here that from this point onwards, each of these groups of buses will be referred to as a "server" from the point of view of queueing of passengers at stations (an aspect covered in subsection 4.4 or in another context simply as a packet.

Then, if p is total number of packets available for service and n^{ℓ} is the number of packets assigned to line ℓ it will be verified that

$$
\sum_{\ell \in L} n^{\ell} \leq p, \; n^{\ell} \geq 0, n_{\ell} \in Z, \; \ell \in L \tag{8}
$$

4.2 Entering and exiting operations at stations

Operations at stations can be analyzed by queueing theory and in this paper a simplified approach is made. In order to this factor may not affect significatively the performance of the system, a set of constraints are developed resulting in a limitation on the total input flow of packets at stations.

A maximum time $\hat{w}_b^{q0}$ will be adopted for staying in queue at space 0 also a maximum time $\hat{w}_b^1$ for staying at queue of space 1.

Blocking time at stations may be produced because total input flow exceeds time required by packets for exiting the station. Also, because spaces 0 and 1 are limited, blocking may arise from space 1 backwards to space 0 because packets cannot leave berths.

Assume that the time required for operation at a station by a packet is κ_b. κ_b is comprised by maneuver times V3(b) and V3(c) described in section 3. Additionally it will be assumed that time V3(c) is constant. That is to say that the packet stays a fixed time independently of PST actually required for boarding/alighting of passengers (this is equivalent to say that a holding strategy is used at each of the stations). κ_b will depend on the number of vehicles in a packet, berths's capacity, maximum capacity amongst the buses which compose a packet

Let us consider:

- L^0 Maximum number of packets allowed queuing at the entrance in order to access boarding berths + maximum number of packets servicing passengers at berths. (space 0 in the following)
- L^1 Maximum number of packets allowed at the exit queueing for leaving the station. (space 1 in the following)
- η^0 , η^1 Coefficients at 95% of probability for maximum occupancy at spaces 0 and 1 respectively. These factors are in fact the ratio between average and maximum occupancy. A value of 3 has been adopted in the computational experiences of the model.

Applying Little's formula to both spaces 0 and 1 of the station the following constraints will be obtained:

$$\frac{(\kappa_b + \hat{w}^{q0})}{H} \sum_{\lambda \in L_b} z^\lambda \le \frac{L^0}{\eta^0}, \; b \in \hat{N}_G \tag{9}$$

$$\frac{\hat{w}^1}{H} \sum_{\lambda \in L_b} z^\lambda \le \frac{L^1}{\eta^1}, \; b \in \hat{N}_G \tag{10}$$

In summary:

$$\sum_{\lambda \in L_b} z^\lambda \le \hat{Z}_b, \; b \in \hat{N}_G \tag{11}$$

being $\hat{Z}_b = Hmin\left\{ \frac{L^0}{\eta^0 (\kappa_b + \hat{w}^{q0})}, \frac{L^1}{\eta^1 \hat{w}^1} \right\}$

4.2.1 Operation on berths

Assume that the grouping of available buses into packets has been solved satisfactorily. Assume also that n^ℓ have been assigned to line ℓ **.**

By $z^\ell \in Z$ it will be denoted the total number of services (runs) that have to be carried out on line ℓ. Then, the loading factor ρ_b for station b will be given by:

$$\rho_b = \frac{Arrival\ of\ packets\ per\ unit\ of\ time}{Maximum\ number\ of\ services} \tag{12}$$

$$Arrival\ of\ packets\ per\ unit\ of\ time\ at\ station\ b = \frac{1}{H}\sum_{\ell\in L_b} z^{\ell} \tag{13}$$

$$Maximum\ number\ of\ services\ per\ unit\ of\ time\ at\ station\ b = \frac{s_b}{\kappa_b} \tag{14}$$

Then:

$$\rho_b = \frac{\kappa_b}{s_b H}\sum_{\ell\in L_b} z^{\ell} \tag{15}$$

The number of services z^{ℓ} and the number of packets assigned to a line must verify:

$$n^{\ell} \geq \frac{z^{\ell}}{H}\left(C_{\ell}^{0} + \sum_{b\in\Pi_{\ell}} \left(\kappa_b + \hat{w}_b^{1} + w_{\ell b}^{B}(\rho_b)\right)\right),\ n^{\ell}, z^{\ell} \in Z,\ \ell \in L \tag{16}$$

C_{ℓ}^{0} is the cycle length for line ℓ without delays and $w_{\ell b}^{B}(\rho_b)$ is the average delay at station b as a function of the loading factor ρ_b (afluencia de convoyes en la parada). If a bound $\hat{w}_b^{q0}$ is taken on $w_{\ell b}^{B}(\rho_b)$, then previous relationship (16) can be rewritten as

$$\frac{Hn^{\ell}}{C_{\ell}^{1}} \geq z^{\ell},\ \ \ell \in L \tag{17}$$

where C_{ℓ}^{1}:

$$C_{\ell}^{1} = C_{\ell}^{0} + \sum_{b\in\Pi_{\ell}} \left(\kappa_b + \hat{w}_b^{1} + \hat{w}_b^{q0}\right) \tag{18}$$

4.3 Link travel time functions on the expanded graph

Because of graph's structure described in figure 4 and accordingly to the distinct operational times described at the beginni9ng of section 3, link cost (travel time) functions will be the following ones:

$$T_a(v) = \begin{cases} t_a(v) = t_a^{P}, & a=(i,j), i\in N_G,\ j\in N_G & (pedesterian) \\ t_a(v) = t_a^{B}, & a=(i,j), i\notin N_G,\ j\notin N_G & (in-vehicle) \\ t_a(v) = t_a^{E}, & a=(i,j), i\notin N_G,\ j\in N_G & (alighting) \\ t_a(v) = w_b^{P\ell}\left(\dfrac{v_a}{cz^{\ell} - v_{x(a)}}\right), & \begin{array}{l} a=(i,j), i\in N_G,\ j\notin N_G \\ a=a(\ell,b), b\in\Pi_{\ell}, \ell\in L, \end{array} & (boarding) \\ t_{x(a)}(v) = \hat{w}_b^{q0} + \hat{w}_b^{1} + \kappa_b, & \begin{array}{l} a=(i,j), i\in N_G,\ j\notin N_G \\ a=a(\ell,b), b\in\Pi_{\ell}, \ell\in L, \end{array} & (on-boardwaiting) \end{cases} \tag{19}$$

Also, as a function of decision variables n^{ℓ}, z^{ℓ}, link cost function for link $a=(i,j)$, $i\in N_G,\ j\notin N_G$ corresponding to passengers' waiting time t_a at a station $b\in A_{\ell}$ until boarding line $\ell, \in L, a = a(\ell,b)$ can be expressed as:

$$t_a(v) = \left(\frac{h}{2z^\ell} + \frac{z^\ell \sigma_{\ell b}^2}{2H}\right) \cdot f_a\left(\frac{v_a}{cz^\ell - v_{x(a)}}\right), \begin{array}{l} a = (i,j), i \in N_G, \, j \notin N_G \\ a = a(\ell, b), b \in \Pi_\ell, \, \ell \in L, \end{array} \tag{20}$$

4.4 Approximation of total waiting time of passengers at a station

In this section models for the waiting time of passengers at stations are investigated and, as the model design that is going to be presented is of the system optimum type, characteristics of total waiting time of passengers queueing at stations needs to be investigated. It is also very relevant to obtain relationships and constraints in order to be included in a formulation of the linear-integer programming type.

A basic hypothesis on behaviour of passengers at stations is that waiting queues are specific for each of the lines and that a strategies model [18] is not followed. Because a high degree of congestion is to be expected on the system, also a high occupancy of the stations is to be expected, making difficult for passengers to board on the first arriving bus within a set of candidate lines. That is to say, if L_b is the set of line stopping at station $b \in N_G$, then there are $|L_b|$ passenger queues, each one for a line $\ell \in L_b$. If passengers might follow a strategies model, then a congested user equilibrium transit assignment, such as the one in [5] might be used.

It is commonly accepted that passengers' waiting at a bus stop can be modeled by means of a queuing process with the following characteristics:

• bulk service. i.e., passengers' arrivals is on a one to one basis, but there is batch-service for each bus arrival at the stop. Usually the total time used for boarding is much smaller than the bus interarrival time at the station and can be neglected in a first approximation.

• the number of passengers that can be allocated at each arriving bus a random variable which is independent from the number of passengers waiting at the moment of arrival.

• Random server's interarrival time with a generic distribution of probability.

Passenger's arrival at stations has been assumed to be poissonian and in the context of queueing theory the more suitable model seems to be an $M/M^{[Y]}/1$. Using simulation several probability distribution models have been examined for server's inter-arrival time. In general it has been found that, for the case of a station with a single berth, queueing model's response is similar to the one of the well known $M/M^{[Y]}/1$. For this queueing model the average waiting time per passenger φ for a given bus line at a station is given as a function of the loading factor of the queue ρ and the average number of services z per period of time received by the station as

$$\varphi = \varphi(\rho, z) = \frac{h}{z}\xi(\rho) \tag{21}$$

$Z(z)$ is then the waiting time at the bus-stop per service and passenger in an uncongested operation, whereas $\xi(\rho)$ plays the role of an augmentative factor due to the loading factor or traffic factor ρ of the queue, which can be expressed as a ratio between the passenger's arrival rate and the rate of passengers that servers can alleviate from the station.

Taking into account the topology of the expanded network shown in figure 2, the loading factor ρ will be expressed as a function of flows as

$$\rho = \frac{v_a}{\eta - v_x} \tag{22}$$

and accordingly, total passenger waiting time boarding at a line will be expressed as:

$$\zeta = v_a Z(z)\xi(\rho) = v_a Z(z)\xi\left(\frac{v_a}{\eta - v_x}\right) = Z(z)\phi(v_a, v_x) \tag{23}$$

where η is the total alighting capacity of the bus line during the horizon of H minutes and ϕ is the mean overall waiting time for passengers at a station per period of service at that station. Now if the following hypothesis is verified,

Hypothesis. Function $\xi(\cdot)$ is non-decreasing and convex in $[0,1]$ and

$$\xi(0) = 1,\ \ell im_{\rho \to 1}\xi(\rho) = \infty \tag{24}$$

Then, under these conditions it can be shown that function ϕ is convex in (v_a, v_x) and that, accordingly, it can be approximated by a finite set of cutting planes to the surface $(\phi(v_a, v_x), v_a, v_x)$ in $\Re^3$ and so, it can be easily incorporated into an optimization model. So, an approximation can be made to ϕ on $B = \{\,(v_a, v_x) \in \Re^2 \mid v_a + v_x \le \eta, v_a \ge 0, v_x \ge 0\}$ $(\Rightarrow \rho \le 1)$ by means of its values and gradients at a set S of n_ϕ points :

$$S = \left\{(v_a^{(k}, 0) \in \Re^2 \mid v_a^{(k} \in [0, \eta[\ , 0 \le k \le n_\phi - 1\right\} \tag{25}$$

If $\phi_a^{(k} = \phi(v_a^{(k}, 0)$, $\hat{\phi}_a^{(k} = \nabla_{v_a}\phi(v_a^{(k}, 0)$, $\hat{\phi}_x^{(k} = \nabla_{v_x}\phi(v_a^{(k}, 0)$ with $(v_a^{(k}, 0) \in S$, then approximation to ϕ on B can be made by means of the convex function ψ:

$$\psi(v_a, v_x) = max_{0 \le k \le n_\phi - 1}\left\{\phi_a^{(k} + \hat{\phi}_a^{(k}(v_a - v_a^{(k}) + \hat{\phi}_x^{(k} v_x\right\} \tag{26}$$

more explicitly:

$$\psi(v_a, v_x) = max_{0 \le k \le n_\phi - 1}\left\{v_a^{(k}\xi(\rho_k) + \big(\xi(\rho_k) + \rho_k \xi'(\rho_k)\big)(v_a - v_a^{(k}) + \rho_k^2 \xi'(\rho_k) v_x\right\} \tag{27}$$

being $\rho_k = \dfrac{v_a^{(k}}{\eta}$. If $v_a^{(k} = \eta k/(n_\phi - 1), k = 0,1,2,\dots, n_\phi - 1$, then there follows that

$$\rho_k = \frac{k}{n_\phi - 1}, k = 0,1,2,\dots, n_\phi - 1$$

4.4.1 Application of previous approximation with a unique class of servers

In the model, total alighting capacity for an amount of z services will be given by $\eta = cz^\ell$, where c is the average maximum capacity for the set of units making up a server and z^ℓ the number of services received at the station during the period of h minutes. For a link $a = a(\ell, b), b \in L_b, \ell \in L$

$$\begin{aligned} v_a^{(k}\xi(\rho_k)+\big(\xi(\rho_k)+\rho_k\xi'(\rho_k)\big)(v_a-v_a^{(k})+\rho_k^2\xi'(\rho_k)v_x &= \\ -\rho_k\xi'(\rho_k)v_a^{(k}+(\xi(\rho_k)+\rho_k\xi'(\rho_k))v_a+\rho_k^2\xi'(\rho_k)v_x &= \\ -\rho_k^2\xi'(\rho_k)cz^\ell+\rho_k^2\xi'(\rho_k)v_x+(\xi(\rho_k)+\rho_k\xi'(\rho_k))v_a &= \\ -c\tilde{\beta}_k z^\ell+\tilde{\beta}_k v_x+\tilde{\gamma}_k v_a & \end{aligned} \quad (0\le k\le n_\phi-1) \tag{28}$$

And thus,

$$\zeta = Z(z)\phi(v_a,v_x) \approx Z(z)\psi(v_a,v_x) = Z(z)max_{0\le k\le n_\phi-1}\left\{-c\tilde{\beta}_k z^\ell+\tilde{\beta}_k v_x+\tilde{\gamma}_k v_a\right\} \tag{29}$$

Accordingly, total waiting time ζ at a station could be included in a mathematical programming formulation as the set of constraints:

$$Z(z)(-c\tilde{\beta}_k z^\ell+\tilde{\beta}_k v_x+\tilde{\gamma}_k v_a)\le\zeta,\ \ 0\le k\le n_\phi-1 \tag{30}$$

In case of $Z(z)=h/z$:

$$\tilde{\beta}_{a,k}\frac{v_{x(a)}}{z^\ell}+\tilde{\gamma}_{a,k}\frac{v_a}{z^\ell}\le\frac{\zeta_a}{h}+c^\ell\tilde{\beta}_{a,k},\ a=a(\ell,b), b\in\Pi_\ell,\ \ell\in L, 0\le k\le n_\phi-1 \tag{31}$$

coefficients $\tilde{\beta}_{a,k},\tilde{\gamma}_{a,k}$ in previous constraints are then:

$$\tilde{\beta}_{a,k}=\rho_{a,k}^2\xi'(\rho_{a,k}),\ \ a=a(\ell,b), b\in\Pi_\ell,\ \ell\in L, 0\le k\le n_\phi-1 \tag{32}$$

$$\tilde{\gamma}_{a,k}=\xi(\rho_{a,k})+\rho_{a,k}\xi'(\rho_{a,k}),\ \ a=a(\ell,b), b\in\Pi_\ell,\ \ell\in L, 0\le k\le n_\phi-1 \tag{33}$$

It must be noticed that ratio v_a/z^ℓ is the average number of passengers boarding at each service and that $v_{x(a)}/z^\ell$ is the average number of on board passengers arriving at each service which do not alight from the server. Thus, if maximum capacity of servers of line ℓ is c^ℓ passengers, a discretization can be performed on the values of the ratio v_a/z^ℓ.

4.4.2 Queueing time per passenger at stations

Accordingly to the formulation in previous subsection 4.4 queueing time per passenger at station in order to board on a vehicle of line ℓ will be given by:

$$\frac{\zeta_{a(\ell,b)}}{v_{a(\ell,b)}},\ \ (v_{a(\ell,b)}>0) \tag{34}$$

5 Modelling bus delay at stations

The $GI/G/s$ queueing model will be adopted for servers (packets) entering a station. For these queueing systems Allen-Cunnen's formula provides very good approximations for average queueing time. In queuing theory notation and terminology, if w_q is the queuing time per passenger, then

$$E[w_q]=W_q\approx\frac{C(s,\theta)(C_\tau^2+C_x^2)}{2s\mu(1-\rho)} \tag{35}$$

where $\rho=\lambda/\mu$ is the queue's loading factor, i.e. the ratio between the clients arrival rate λ and μ, the service rate of an individual server of the service system, s is the number of servers, C_x

and C_τ are the coefficients of variation of the interarrival times τ for clients and the service time x respectively. $C(s,\theta)$ is the so called Erlang's function:

$$C(s,\theta) = P_{M/M/s}(N \geq s) = \frac{\dfrac{\theta^s}{s!(1-\rho)}}{\sum_{\ell=0}^{s-1} \dfrac{\theta^\ell}{\ell!} + \dfrac{\theta^s}{s!(1-\rho)}}, \ being\ \theta = \frac{\lambda}{\mu} \tag{36}$$

Typically in queueing theory, delay formulas as a function of the loading factor are convex functions. So it is possible to approximate them as:

$$W_q(\rho) \approx max_{1\leq v\leq m}\{a^v + d^v \rho\} \qquad (37)(38)$$

The following figure 3 shows one such approximation in terms of the normalized delay W_q':

$$W_q' = \frac{\mu W_q}{C_\tau^2 + C_x^2} = \frac{C(s, s\rho)}{2s(1-\rho)} = W_q'(\rho)$$

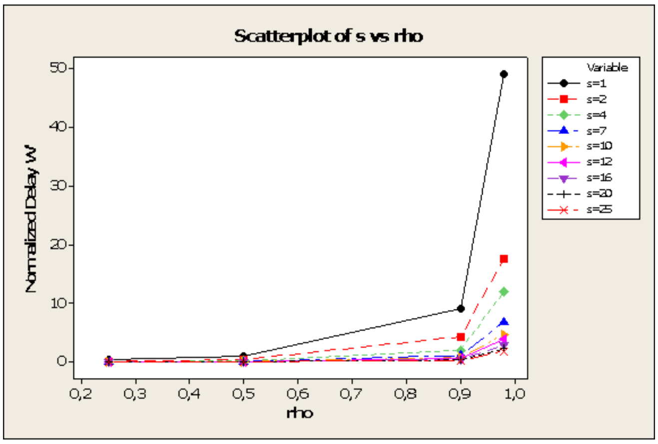

Figure 3. Approximation of normalized delay W_q' for the queue GI/G/s for several number of servers s using piecewise linear approximation (37).

6 The design model in terms of integer-linear programming

Total waiting time $= v^{\mathrm{T}}T(v)$ spent by passengers at stations must play a main role in a system optimum formulation. Taking into account all components of $T(v)$ appearing in (19), the term $v^{\mathrm{T}}T(v)$ will be:

$$v^{\mathrm{T}}T(v) = \sum_{a\in A_G} t_a^P v_a + \sum_{\ell\in L}\left(\sum_{a\in A_\ell} t_a^B v_a + \sum_{b\in\Pi_\ell} v_{a(\ell,b)}\xi_{a(\ell,b)}\left(\frac{v_{a(\ell,b)}}{cz^\ell - v_{x(a(\ell,b))}}\right)\right) \tag{39}$$

Notice that (39) contains terms of the form $v\xi$ and thus they can approximated by means of (27). An equivalent approximation for $v_a\xi$ can be used tom set bounds on the passengers occupancy of stations. Assume that $\hat{N}_b^{pax}$ is the maximum number of passengers that can be allocated at a station $b\in \hat{N}_G$. The total length of passenger queues waiting for packets should not exceed this amount at least in a very high percentage of the horizon time. Then applying Little's formula:

$$\frac{v_{a(\ell,b)}}{h}\xi_{a(\ell,b)} = Average\ number\ of\ \ passengers\ in\ queue\ a(\ell,b) \tag{40}$$

And taking into account passenger queues for each of the lines stopping at station b :

$$\sum_{\ell\in L_b}\zeta_{a(\ell,b)} \leq \frac{h}{\eta_b}\hat{N}_b^{pax},\ b\in \hat{N}_G \tag{41}$$

where η_b is the ratio between average occupancy and the maximum occupancy at, say, 95% of the time.

6.1 Linear integer programming formulation of the design model

Putting together all constraints developed in previous sections and adopting a system optimum point of view, the design model expressed as an integer linear mathematical programming problem is as follows. A summary of the notations used is provided for easy of reading.

n^{ℓ} = number of packets assigned to line ℓ .

z^{ℓ} = number of services assigned to line ℓ .

v_a = passenger flow at link a in the expanded graph; if $a = (b, j), b\in \Pi_{\ell}$, i.e. a is a boarding link then by $x(a)$ and $y(a)$ are denoted in-vehicle waiting links and alighting links of the expanded graph network. (see figure 2). When convenient boarding links in the expanded graph network which are located at station b in order to board on line ℓ are denoted by $a(\ell,b)$

C_{ℓ}^{1} = cycle time of line ℓ

p = number of available packets

h = planning horizon in minutes

t_a^P,t_a^B = travel time for link a of the expanded network for pedestrian and in-vehicle travel time in case of a being a pedestrian link or a link of the expanded transit network in bus mode.

c = average capacity (pax) of packets.

$\hat{f}$ = maximum frequency admissible on any line.

s_b = number of berths at station b .

κ_b = maneuver times V3(b) + V3(c) = Maneuver times for entering/exiting berths + PST

$\hat{w}_b^{q0}$ = maximum queueing time for packets in order to access berths of station b .

$\hat{N}_b^{pax}$ = maximum number of passengers that can be allocated in station b .

$\hat{Z}_b,\eta_b$ = see section **¡Error! No se encuentra el origen de la referencia.**.

ζ^{ℓ} = cost of assigning a packet to line ℓ .

γ^{ℓ} = operational cost of a service at line ℓ

θ = social cost of passenger's travel time.

It must be noticed that n^{ℓ},z^{ℓ} are the main decision variables of the optimization problem. All other decision variables, $\delta_i^a,\delta_i^{x(a)},\rho_b,u_a,\zeta_a$, play an auxiliary role.

Constraints I establish the relationships amongst the cycle of the lines, the vehicles assigned to them and the number of services; constraints II establish that a maximum frequency should not be exceeded in order to avoid bus "bunching"; constraints III establish the assignment of units to the lines; constraints IV, V, VI limit the delay of buses at stations when queueing for access to boarding/alighting berths; constraints VII, VIII modeli the delay of passengers at the stops or stations; constraints IX are the formulation of balance equations in a multicommodity flows network; constraints X limit the frequencies of entrance of buses in stations, so that queues are not created at the entrance or blocking by spillback is created; finally, XI constraints establish limits for the average occupation for the number of passengers in the stations, as they present a physical limited capacity.

$$
\begin{array}{lll}
 & Min_{n,z,v,\zeta} & \sum_{\ell\in L}(\varsigma^{\ell} n^{\ell} + \gamma^{\ell} z^{\ell}) + \theta\sum_{a\in A_G} t_a^P v_a + \theta\sum_{\ell\in L}\left(\sum_{a\in A_\ell} t_a^B v_a + \sum_{b\in\Pi_\ell}\zeta_{a(\ell,b)}\right) \\
I & s.a: & hn^{\ell} \geq z^{\ell} C_{\ell}^{1},\ \ell\in L \\
II & & 0\leq z^{\ell} \leq \hat{f}\cdot h,\ z^{\ell}\in Z,\ \ell\in L \\
III & [SO-1] & \sum_{\ell\in L} n^{\ell} \leq p,\ n^{\ell}\geq 0, n^{\ell}\in Z,\ \ell\in L \\
IV & & \rho_b = \frac{\kappa_b}{s_b h}\left(\sum_{\ell\in L_b} z^{\ell}\right),\ b\in \hat{N}_G \\
V & & 0\leq \rho_b \leq 1-\varepsilon,\ b\in\hat{N}_G \\
VI & & a_b^{\nu} + d_b^{\nu}\rho_b \leq \hat{w}_b,\ 1\leq\nu\leq n_w, b\in\hat{N}_G \\
a = a(\ell,b), & b\in\Pi_\ell, \ell\in L & in\ VII, VIII, XI: \\
VII-1 & & \tilde{\beta}_{a,k} u_{x(a)} + \tilde{\gamma}_{a,k} u_a \leq \frac{\zeta_a}{h} + c\tilde{\beta}_{a,k},\ 0\leq k\leq n_\phi - 1 \\
VII-2x & & z^{\ell} i - (1-\delta_i^{x(a)})M \leq v_{x(a)} \leq z^{\ell}(i+1) + (1-\delta_i^{x(a)})M,\ i = 0,1,2,...,c-1 \\
VII-3x & & u_{x(a)} = \frac{1}{2}\sum_{i=0}^{c-1}(2i+1)\delta_i^{x(a)}, \\
VII-4x & & \sum_{i=0}^{c-1}\delta_i^{x(a)} = 1,\ \delta_i^{x(a)}\in\{0,1\} \\
VII-2a & & z^{\ell} i - (1-\delta_i^{a})M \leq v_a \leq z^{\ell}(i+1) + (1-\delta_i^{a})M,\ i = 0,1,2,...,c-1 \\
VII-3a & & u_a = \frac{1}{2}\sum_{i=0}^{c-1}(2i+1)\delta_i^{a}, \\
VII-4a & & \sum_{i=0}^{c-1}\delta_i^{a} = 1,\ \delta_i^{a}\in\{0,1\} \\
VIII & & v_a + v_{x(a)} \leq cz^{\ell} \\
IX & & v\in V \\
X & & \sum_{\ell\in L_b} z^{\ell} \leq \hat{Z}_b, b\in\hat{N}_G \\
XI & & \sum_{\ell\in L_b}\zeta_{a(\ell,b)} \leq \frac{h}{\eta_b}\hat{N}_b^{pax},\ b\in\hat{N}_G
\end{array}
\tag{42}
$$

7 Application of the model

During the meeting presentation the application of the previous model to a test network reproducing a corridor in Madrid (Túnel de la Risa) between the stations of Atocha, Recoletos, Nuevos Ministerios and Chamartín.

	At	*Re*	*NM*	*Ch*	*TotalOr.*	*pax/min*
At	0	2.011	22.097	368	24.476	135,98
Re	170	0	3.066	230	3.466	19,25
NM	4.386	150	0	170	4.706	26,14
Ch	2.504	150	2.438	0	5.092	28,28
TotalDest.	7.060	2.311	27.601	768	37.740	–
pax/min	39,22	12,84	153,34	4,26	–	–

O-D Trip matrix (station-to-station) during period $H = 180$ minutes. Last row and column are average rates for arrivals and departures per minute at stations. Some of them are rather high.

In the following figure 4, the line setting is shown (left). Also in figure 4 (right) the expanded network is outlined accordingly to the description in section 2

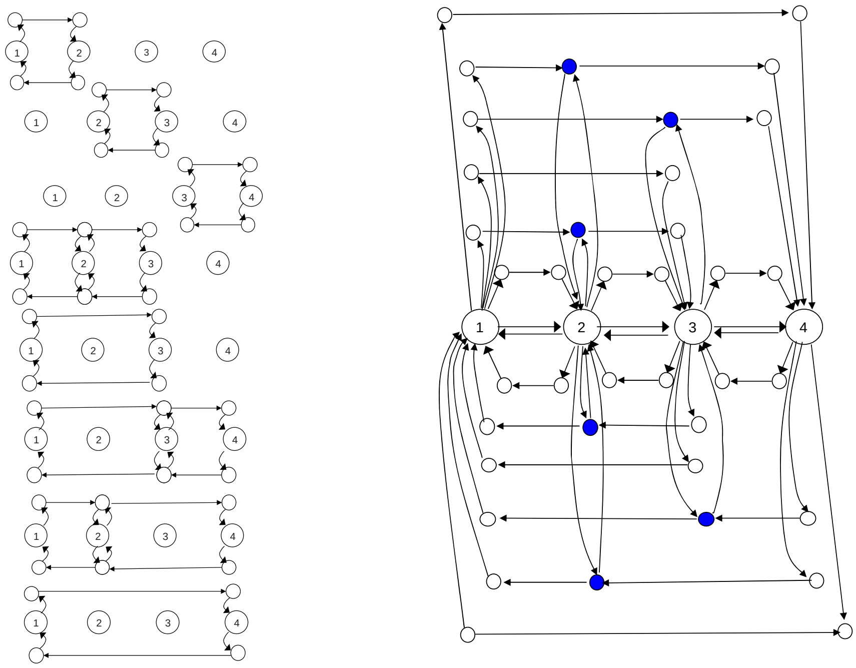

Figure 4. Network links for lines L1, L2, L3, L4, L5, L9, L10, L11, shown isolatedly.(left). Expanded graph for lines L1, L2, L3, L4, L5, L9, L10, L11.(right)

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