APPLICATION OF THE THEORY OF CHAOS IN AN URBAN TRANSPORT NETWORK

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ABSTRACT

Chaos is a phenomenon that may occur in systems which are dynamic, nonlinear, sensitive to their parameters, sensitive to their initial conditions, or possess self-similarity (Disbro and Frame, 1989; Dendrinos, 1990; Moon, 1992; Cambel, 1993; Chen and Leung, 1998; and Kapitanik, 2000). This study has examined the application of the theory of chaos in an urban transport network.

Starting with speed-flow-density relationships and unstable flow concept in basic freeway sections, analogous relationship has been devised at network level for speed-throughput of the network. Network throughput is defined as the sum of O/D trips which are finished satisfactorily. Simulation runs have been made to produce network speed-throughput relationship similar to the ones presented by Greenshields and others for a basic freeway section. A real network has been used for this purpose. A simulation process has been devised which inputs demand into a real network and measures the throughput from the network at various times and conditions. Random variation has been considered both for demand and transport network. Random variation of demand in this study comes (a) from the variation in the starting times of trips with trip purposes such as work and school which should happen and end before certain times, and (b) from the variation in the starting time as well as total daily demand of other trips with more flexible trip purposes such as shopping, recreation and social, and personal business. Variation in transport network comes from the closure of selected links due to accidents. For this purpose the probability of an accident occurrence in each link has been defined.

A Lyapunov exponent measure has been devised for a series of equilibrium maximum throughputs to investigate whether chaos occur in the network at a certain transport demand level. The exponent was suggestive of this phenomenon, and ways to avoid it were discussed by defining a population capacity for the city, or expanding the network capacity.

Keywords: : *Urban Transportation, Chaos, Greenshields Functions*

1. INTRODUCTION

Traffic on an urban transport network can be considered as a dynamic system. Some recent studies using chaos theory to describe the traffic flow phenomena have attracted the interest of many researchers and practitioners (Li and Gao, 2004).

Van Zuylen et al. (1999) discussed the implications of human behavior, chaos and unpredictability for urban and transportation planning and forecasting. Safanov et al. (2002) showed that chaotic behavior in traffic flow can be due to the delays in human reaction. Weidlich (2000) demonstrated how random-utility-based models of relatively simple social behaviors produced chaotic behavior. Li and Gao (2004) study the characteristic behavior of traffic flow at low and intermediate density values by the nonlinear time series analysis approach.

This work offers a detailed application of chaos to an urban transport network. The purpose of this paper is to model an urban area in terms of the viability of its transport system to back the city as a life line. It, first, revisits an old road level observation in section 2. Sections 3 to 5 discuss the famous speed-density-flow relationships at network level, and specify a speed-throughput relationship. Section 6 tries to provide simple description of theory of chaos and its measures, which is later applied on the real network of the case under study in section 7. Section 8 takes advantage of this concept to define a limit to the growth of the city, and introduce ways to relax the constraints exerted upon the city and concludes the paper.

2. GREENSHIELDS MODEL REVISITED

Greenshields (1934) proposed a linear speed-density model based on several field studies:

$$v = v_0 - \frac{v_0}{D_j} \cdot D \qquad \text{(speed-density relationship)} \qquad (1)$$

where v is the (space) mean speed (e.g., in km/hr), v0 the free flow speed, D the flow density (e.g., in veh/km), and Dj the jam density. For q, as the (rate of) flow (e.g., in veh/hr), a basic speed-flow-density relationship for a basic freeway section under ideal conditions may be stated as:

$$q = v.D \qquad (2)$$

Substituting D from Equation 1 into Equation 2 results:

$$q = D_j v - \frac{D_j}{v_0} \cdot v^2 \qquad \text{(flow-speed relationship)} \qquad (3)$$

Computing v from Equation 2 and substituting in Equation 1 results:

$$q = v_0 D - \frac{v_0}{D_j} \cdot D^2 \qquad \text{(flow-density relationship)} \qquad (4)$$

McShane and Roess (1990) report a study by Drake et al. (1967) (see McShane and Roess (1990)) in calibrating the Greenshields' model (as well as those of others) for speed-flow-density relationship for a basic freeway section data. Figure 1 shows the data and two sets of the calibrated models for the data in that study. For the Greenshields model the relationships and the parameters are calibrated as follow:

$$v = 58.6 - 0.465\,D \qquad (5)$$

$$q = 126.02v - 2.15\,v^2 \qquad (6)$$

$$q = 58.6\,D - 0.465\,D^2 \qquad (7)$$

v0 =58.6 mph
Dj =126.02 vpm
$q_{max} \cong 1846.6\ \text{vph}$ at $v = 29.31\ \text{mph}$

In this figure, there is a clear dislocation, or discontinuity, of data points at the point where the flow rate reaches its maximum. This flow rate is, also, referred to as "unstable flow" (McShane and Roess (1990)). This break-down in data points has persuaded the analysts to present a two-segment model, one for each side of the discontinuity point, as shown in Figure 2 (Ellis (1964), as referenced in McShane and Roess (1990)). McShane and Roess (1990) report other models in this category in order to explain the variation in the data better.
There are a few points worth mentioning regarding Figure 1(or 2). First, it is believed that drivers may only sense density and adjust their speeds accordingly. Flow, which is a point measure, may not be sensed by them, and is a product of their behaviour (McShane and Roess 1990). This is why speed-density relationship is the core relationship in the 3 relationships discussed above. Increase in the incoming flow in a freeway section would increase the density of the flow, which in turn reduces the speed. This reduction in the level of service is compensated by the increase in the rate of flow, until a point is reached where flow rate is maximum.

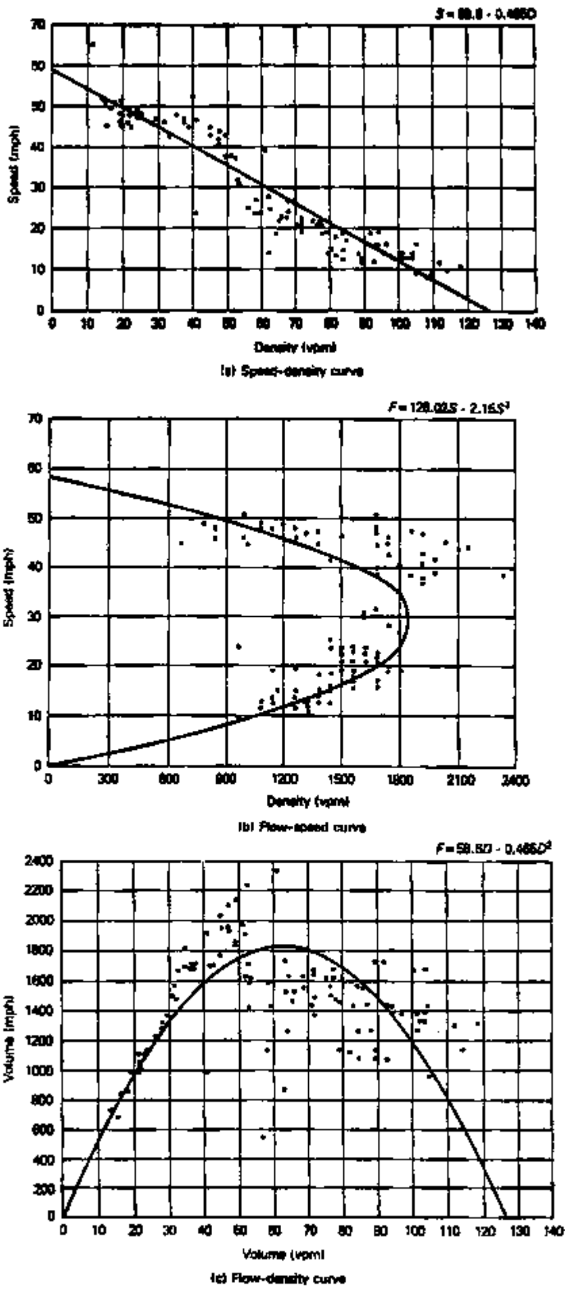


Figure 1. Greenshields' speed-flow-density relationships (from McShane and Roess (1990)).The x-axis in part (b) should read "Volume" and in part (c) "Density."

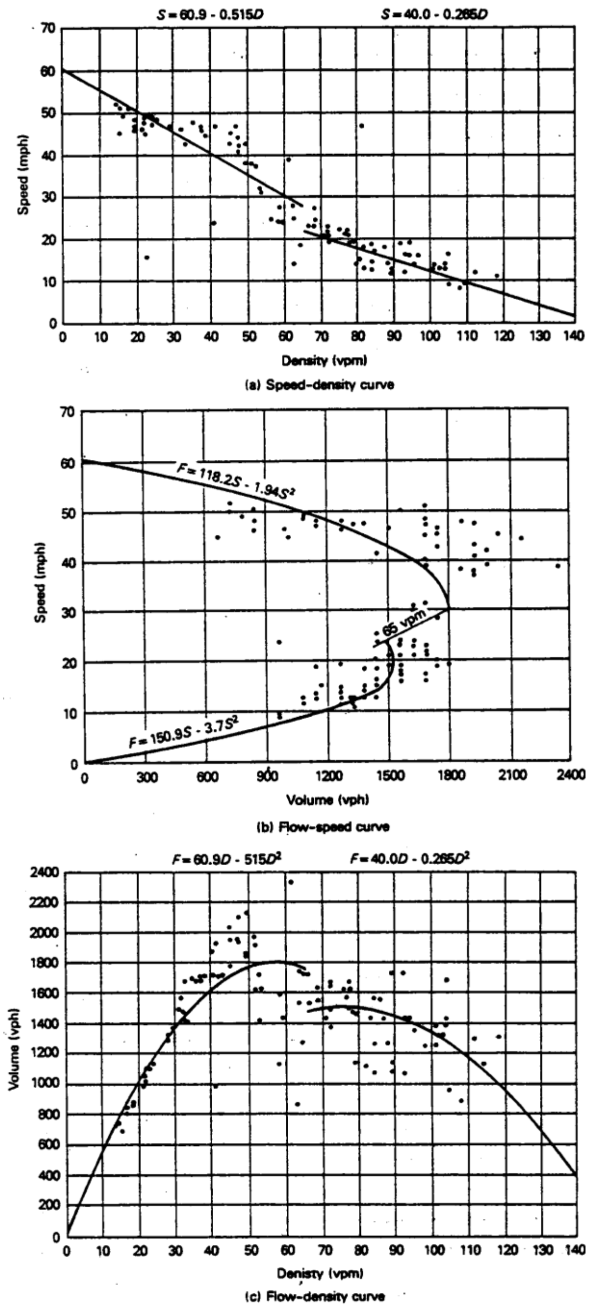


Figure 2. Two segment speed-flow-density relationships (from McShane and Roess (1990)).

Second, at the point of maximum flow rate the process ceases to retain a clear relationship between the state variables speed, density, and flow, so that turbulence occurs. From this point on, increase in density would reduce the speed further while reducing the flow rate as well, both tending to zero.

Third, as may be seen in Figure 2, there are two peaking characteristics exhibited by the process, one showing a higher capacity when the flow is approaching from a stable condition, and the other (lower) capacity is attained when the flow is in a forced condition. Thus, before the freeway breaks down, rate of flow departing the section is higher than when it does break down. This could be an important concept in design of networks: Design a network so as not to let it get into unstable flow.

3. SPEED-DENSITY-FLOW RELATIONSHIPS AT NETWORK LEVEL

In this section an attempt will be made to create a speed-density-flow relationship at network level similar to those at link level (such as in Figure 1 or 2). To do this, a simulation process has been devised which inputs demand into a real network and measures the throughput from the network at various times and conditions.

The network under study is an aggregate version of the network of Mashhad, Iran, comprised of 872 nodes and 1184 links, and 15 super sources and super sinks distributing the traffic of 141 origins and destinations through the network. Figure 3 shows this network.

The demand system constitute the following set of models, all based on an extensive origin/destination (O/D) survey data base conducted in 1994. There are 7 trip production and attraction models, one for each of 7 trip purposes. These models are functions of several policy variables, such as car-ownership, zonal employment level, and school enrolment.

The trip distribution model is a doubly-constrained growth model of the following form:

$$T_{ij}^{p} = \alpha_i^p \beta_j^p O_i^p D_j^p d_{ij}^{\gamma^p} \qquad (8)$$

where T_{ij}^{p} is the total O/D trips from origin i to destination j for trip purpose p; O_i^p and D_j^p are, respectively, the total trip produced from zone i ,and attracted to zone j, for trip purpose p; d_{ij} is the distance from i to j; and α_i^p , β_j^p are two balancing factors and γ^p a model parameter. The model is subject to the following two constraints:

Figure 3. The aggregate network of the city of Mashhad

$$\sum_{j} T_{ij}^{p} = O_{i}^{p} \qquad , \forall i \tag{9}$$

$$\sum_{i} T_{ij}^{p} = D_{i}^{p} \qquad , \forall j \tag{10}$$

where $\sum_{i} O_{i}^{p} = \sum_{j} D_{j}^{p}$. The model is calibrated by successive row-column balancing for each trip purpose, p.

The mode choice model for each trip purpose is of multinomial logit type:

$$p_{ij}^{m,p} = \frac{e^{u_{ij}^{m,p}}}{\sum_{k \in M} e^{u_{ij}^{k,p}}} \tag{11}$$

where $p_{ij}^{m,p}$ is the probability of choosing mode m among the set of alternative modes M in a trip from zone i to zone j for purpose p, and $u_{ij}^{m,p}$ is the utility of mode m for that trip, which is a function of the characteristics of the mode of travel, passenger, origin, destination, etc., as follows:

$$u_{ij}^{m,p} = \sum_{l} \alpha_l^{m,p} X_{ijl} \tag{12}$$

where $\alpha_l^{m,p}$ is a mode specific/generic coefficient of the ith variable X_{ijl} of the utility function for the trip purpose p. The set of modes for any trip purpose include bus, taxi, two-wheeled vehicles, and private car, as well as minibus for the school trips. Examples of variables X_{ijl} are in-vehicle travel time, out-vehicle travel time, number of boardings for bus; car-ownership, and travel time for private automobiles; and two-wheeled vehicle ownership, and O/D distance for two-wheeled vehicles.

The Traffic assignment problem is an algorithm which assigns public transport demand to the respective network based on optimal strategy procedure (Speiss and Florian, 1989), and passenger cars (or the equivalent demands for five other vehicles) according to user equilibrium flow criteria by using a convex programming routine (Sheffi, 1985). The auto assignment takes care of traffic signal settings, turn prohibitions, and 6 volume-delay functions, and the public transport assignment takes into account several vehicle types in the fleet of public transport vehicles.

A rather comprehensive and extensive routine is written in EMME/2 environment (INRO, 1996 to estimate (hourly) demand, assign demand to the network, and estimate the operational characteristics of the network, such as vehicle-kilometers, vehicle-hours, average speed of vehicles, network congestion levels and distribution, fuel consumption, pollution emissions, to name a few, in various categories such as geographical regions, road type, vehicle type, network type (public/private), or demand type (person/vehicle trips) (Aashtiani et al., 1997).

4. SIMULATION OF NETWORK FLOW

The system of concern is the network of the city under study, represented by $N(V, A)$ where V is the set of nodes and A the set of links of network. Given the O/D demand and the network, the distribution of flow over the network is a deterministic procedure, as defined in Section 3, producing a constant flow over the links of the network. In a given day, and for a given network, demand varies over time from 00:00 to 24:00. However, assuming the demand and its distribution over time are given, the flow for an hour, say peak hour, will be known (constant) over the links.

Nevertheless this (hourly) flow distribution is subject to some variation due to small changes in the timing of the demand appearing in the network, and small variation in the network due to, e.g., lane closures because of traffic accidents or road repairs. That is, small variations such as few minutes of early/late arrivals of demand and few lanes/roads closures due to certain events cause changes in the flow distribution of the network. Such random variations of demand and supply would simulate the traffic pattern in the network more realistically in the long run than the constant demand/supply case. This study concerns itself to the patterns of flow in the various time intervals of the day, observed in many days, and tries to analyze the variation of traffic characteristics of the network, such as average speed and throughput of the network over times of observation.

4.1. Random Variation of Demand

Random variation of demand in this study comes (a) from the variation in the starting times of trips with trip purposes such as work and school which should happen and end before certain times (they constitute 36 and 18 percent of total non-return daily trips in the study area, respectively), and (b) from the variation in the starting time as well as total daily demand of other trips with more flexible trip purposes such as shop (9%), recreation and visit (15%), and personal business (8%).

For simplicity, and as a starting exercise, let d^{ks} represent the total demand for travel from origin k to destination s during a given time interval. Suppose that this quantity is distributed randomly with mean $\bar{d}^{ks}$ and standard deviation σ_{ks}. (Suppose σ_{ks}=0.1 $\bar{d}^{ks}$ for computational purposes presented in this paper.) Let,

$$d^{ks} = \bar{d}^{ks} + \sigma_{ks} Z \tag{13}$$

where Z is a standard normal random variable. Then, $d^{ks} \sim N(\bar{d}^{ks}, \sigma_{ks}^2)$. (Banks et al., 1996) present a procedure for generating standard normal random numbers by two independent uniformity distributed random numbers in interval [0,1].)

4.2. Random Variation of Supply

Supply of transportation is a variable quantity for a multitude of reasons, including closures of lanes due to traffic accidents, unauthorized stops and parkings, and bus stops in a moving column of vehicles in a lane (as operational factors); rain falls, water ponds, icy pavement, and low visibility due to fogs or rain (as weather-related factors); and slow moving and unsuitably maneuvering vehicles (as human-related factors). Generally, variation of link capacity may take a value between 0 to 100 percent of the capacity of the link in its ideal condition. However, for simplicity, it is assumed that only accidents affect the link capacity (appreciably), and if it happens in a link it closes the link completely. (Clearly, such assumptions may be relaxed easily.)

Suppose accidents occur in the links of the network independent of each other, proportional to the vehicle-kilometers traveled in those links, as a function of link type (k) as follows:

$$p_a^k = \frac{l_a v_a^k}{\sum_{a \in A} l_a v_a^k} \tag{14}$$

where p_a^k is the probability of an accident occurrence in link a of type k, v_a^k is the passenger car equivalent volume of traffic in link a, and l_a the length of link a.

Suppose, also, that accidents in the network are stochastic events following a Poisson rule with average rate of occurrence λ^k for road type k. Available data for a real network shows that the average rate of occurrence of accidents in 618 km of arterial links is 6 accidents per hour, and this rate for 162 km of expressway links is 2 during 6:00 AM to 12:00 PM of a working day. Then, the probability of n^k accident occurrence in a unit time (1 hour) interval (t=1) may be computed as:

$$P\{N_t^k = n^k\} = \frac{e^{-\lambda^k} (\lambda^k)^{n^k}}{n^k!} \text{ , k=a (arterial), e (expressway)} \qquad (15)$$

where $\{N_t^k; t \geq 0\}$ is a Poisson process representing the accident arrivals of road type k (= a, e). Observations for the case mentioned before show that link closures because of accidents and return of traffic movement to a normal situation takes about 1.0 hour.

Based on the above statements, the following procedure has been devised to simulate the effect of traffic accidents upon the performance of the network: (a) Based on a Monte Carlo simulation (using a practically absolute maximum number of accidents, and Equation 15, the number of accidents for the next 1.0 hour morning peak is estimated for each road type k. (b) Based on another Monte Carlo simulation and equation 14 for each road type k, the link(s) (if any) where accidents occur during the next morning peak hour would be identified, and closed (for that hour). In this study, it is assumed that users of the network become aware of the accident location(s) (e.g., by local media), and change routes so as to avoid afflicted link(s). (This assumption may be relaxed.)

4.3. Network Throughput

Throughput of a system may be envisaged as an output of that system. In an urban network this quantity may be defined as the number of person or vehicle trips that are served satisfactorily by the network per unit time period.

Definition 1. Trips from origin k to destination s are served satisfactorily during time period u if $\frac{t_u^{ks}}{t_0^{ks}} < \tau$, where t_u^{ks} and t_0^{ks} are the time period u actual and free flow (k, s) O/D travel time respectively, and τ a real number representing the level of service expected from the network.

Definition 2. The network throughput in time period u, q_u , is the sum of all O/D trips which are served satisfactorily:

$$q_u = \sum_{(k,s)\in P} d_u^{ks} . z_u^{ks} \qquad (16)$$

where d_u^{ks} is the demand for travel from origin k to destination s during the time period u, and P the set of all O/D pairs, and z_u^{ks} equals 1 if (k, s) O/D trips are served satisfactorily during time period u, and 0 otherwise.

In this study demand rates are defined for 1 hour, and τ is taken as 3.0. That is, if the O/D travel time exceeds 3.0 times the average free flow travel time the demand is considered under critical state (and unsatisfied). To get a feeling of the study network, it may be worth mentioning that the average free flow travel time for a typical O/D trip is about 20-30 minutes during the peak hour period at an average speed of 30 km/hr.

5. Speed-Throughput Relationship of the Network

In this section of the paper a speed-throughput relationship at network level will be presented in correspondence with the speed-flow relationship of Greenshields for a basic freeway section. Suppose the network understudy is operating over time and one is monitoring the state of the network during the morning peak hour. The equilibrium flow in the network is a deterministic function of the demand system $\{d_u^{ks}; \forall(k,s) \in P, u \geq 0\}$ and the network specifications $N(V, A_u)$, where A_u is the set of working links during the time period u. Thus, for given demand and network the flow during time period u is known and constant.

Suppose that the parameters of the problem are changed by a small amount as follows:

> (a) O/D demand starting times change so that the demand for each O/D pair varies about the respective mean normally with a standard deviation about 10% of the mean as an example for the numerical computations.
>
> (b) Traffic accidents in the network occur according to two independent Poisson processes with average rates of 6 and 2 accidents per hour (6 AM to 12 PM) for arterial and expressway links, respectively, causing link closures in the afflicted links for that hour. It is assumed that traffic is aware of the accident locations in each time period.

The system is run 30 times, starting with some low O/D demand values equal to a small fraction of the estimated peak O/D demand, and the O/D demand multiplier is increased to several times the estimated values, and the equilibrium values of the state variables of the system, here average speed (v) and throughput (q), for each (demand, supply) parameter specification are observed. The average speed is computed as the ratio of total veh-km over total veh-hr in the network.

Figure 4 shows the results of these runs in terms of v-q relationship. As may be seen in this figure, a relationship similar to that of v-q relationship for a basic freeway section exists for the network: As demand increases in the network, average speed of the network decreases while increasing the throughput of the network. This happens until an average speed of about 28 km/hr, at which point the flow becomes unstable showing a wide range of throughput. This instability holds as O/D demands rise and bring the average network speed down to about 15.5 km/hr, at which further increase in O/D demand decreases both the average speed and the throughput of the network.

Regression analysis of the observations presents the following two expressions for the two portions of the v-q relationships, less the unstable throughput region:

$$q = -296.8v^2 + 13088.7v\,, \qquad \overline{R}^2 = 0.97 \qquad (17)$$
$$(-114.2) \quad (129.5)$$

$$q = 10322.9v - 50642.9, \qquad \overline{R}^2 = 0.94 \qquad (18)$$
$$(65.6) \quad (-31.3)$$

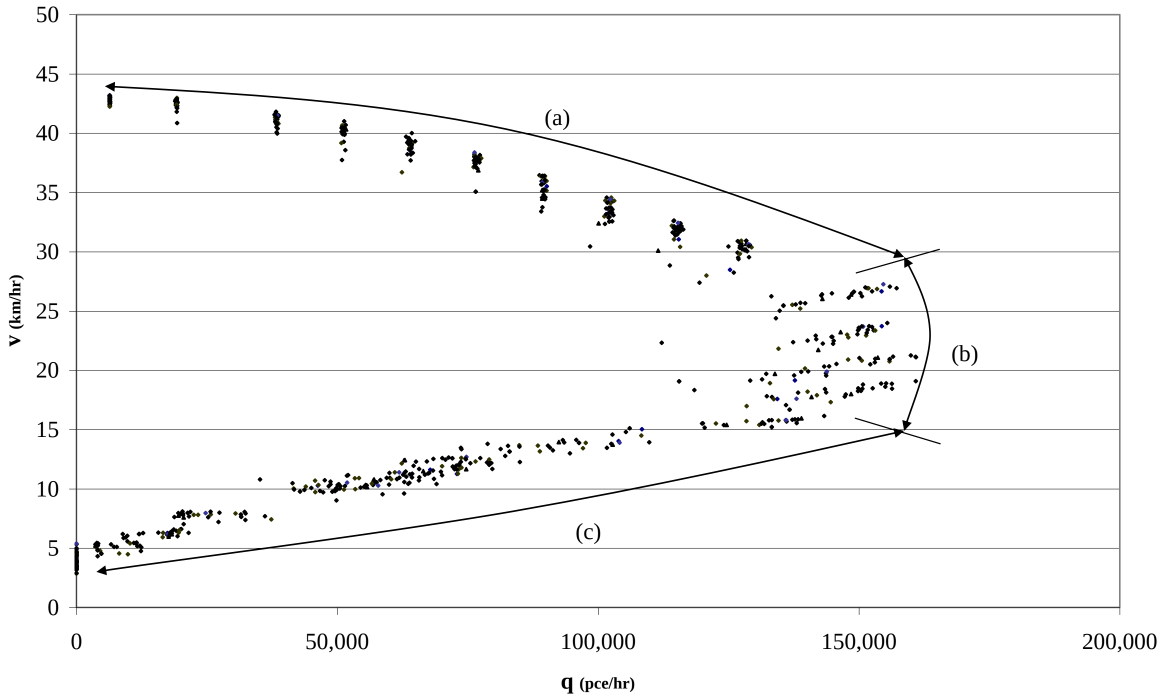
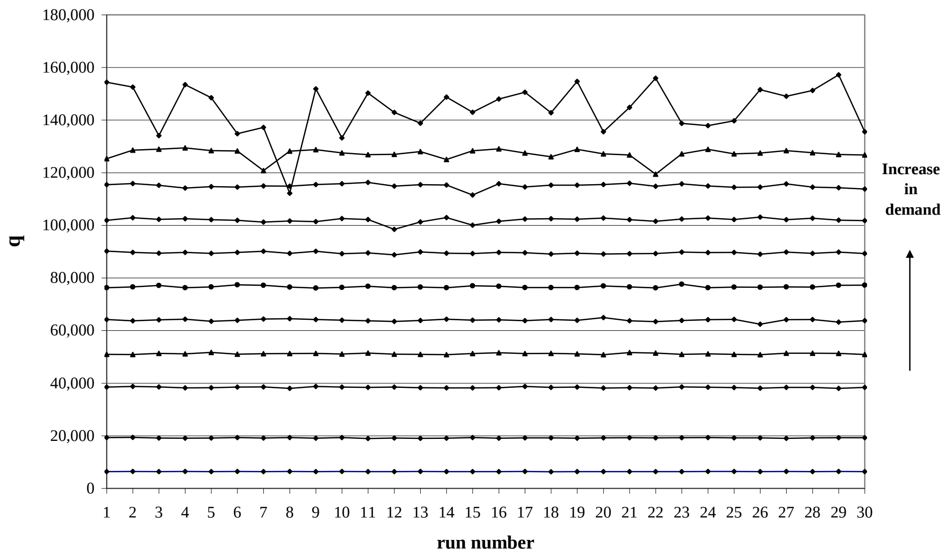


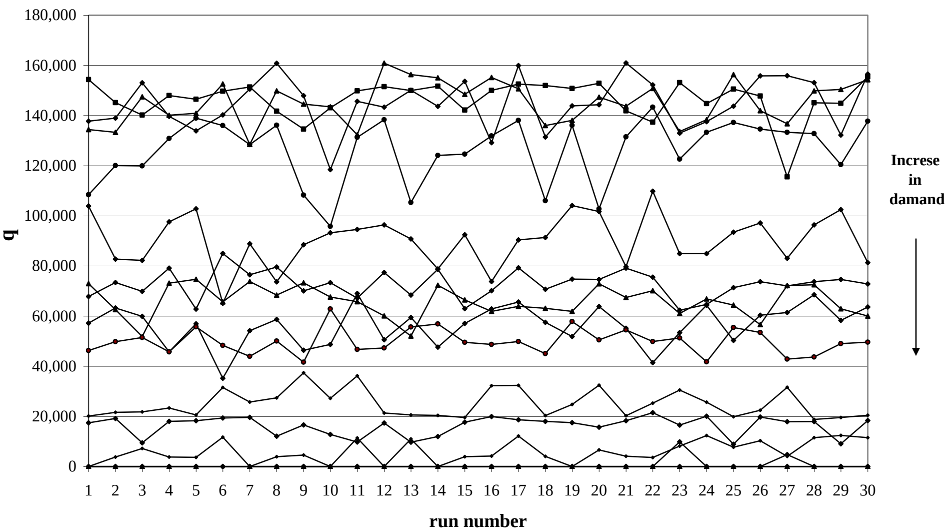
Figure 4- Relationship between the average speed v and throughput q when the network is subject to small variations in demand and supply parameters.

The values of t-statistics in parentheses under estimated parameters show that they are significantly different from 0, and the null hypotheses may be rejected at high confidence levels. Moreover, $\overline{R}^2$ for the estimated relationships show high fitness of the expressions to the observations. It is worth noting that when q=0, $v \cong 5$km/hr, showing that there are links/paths connecting nearby zones in the network that pass flow through themselves with non-zero average speed, despite the fact that demands are not satisfied at all according to Definition 1.

Figure 5.a shows how increase in demand from a small portion of peak hour demand to the highest throughput level in region (a) of Figure 4 changes the starting laminar flow from the beginning (runs) to a turbulent one at the end. Figure 5.b shows the same picture when further increase in O/D demand drags the system into region (b), and then to (c), with highly turbulent states.



(a) region (a)



(b) regions (b) and (c)

Figure 5.- Variation in throughput for various levels of O/D demand values in 30 runs of the network system.

6. CHAOS IN URBAN TRANSPORT NETWORKS

Chaos is a phenomenon that may occur in systems which are dynamic, nonlinear, sensitive to their parameters, sensitive to their initial conditions, or possess self-similarity (Chen and Leung 1998; Cambel 1993, Moon 1992; and Kapitanik, 2000). Schuster and Just (2005), also, present several possible qualitative criteria for chaotic motion: "looking chaotic," exhibiting broadband noise, decaying correlation function, and showing space filling points in Poincare' map, and introduce some quantitative measures to characterize deterministic chaos such as Lyapunov exponent, invariant measure, and correlation function for one-dimensional Poincare' maps, as well as quantitative measures for other cases.

The purpose of this section is to show that the network system of concern in the previous sections exhibits a chaotic behavior when demand passes certain threshold level. To do this, the concept of Lyapunov exponent will be employed, as described below.

6.1. Lyapunov Exponent

Lyapunov Exponent measures the exponential separation in time of the adjacent points which leads to chaotic behavior (when the exponent is positive) (Schuster and Just (2005)). It is a measure of the average separation of adjacent points. Adjacent points are points defining the parameters of the system which are infinitesimally separated from each other. Thus, positive Lyapunov exponent implies average (as becomes clear from the following definition) rate of separation of the states of the system started from those points and run for a long time.

To elaborate further on this concept regarding to the system of traffic flow in an urban network, let $\mathrm{N(V,A)}$ be the network of concern with the set of (k, s) O/D pairs P, having the demand system D: $\{d^{ks};\forall(k,s)\in P\}$. Let, also, be:

ρ : a path in the network ,

ρ^{ks}: the set of path from k to s, $(\mathrm{k,s})\in\mathrm{P}$,

x_ρ^{ks} : the flow in path ρ from k to s, $\rho\in\rho^{\mathrm{ks}}$ and $(\mathrm{k,s})\in\mathrm{P}$,

x_{ij} : the flow in link (i,j), $(i,j)\in A$,

t_{ij} : the travel time in link (i,j), $(i,j)\in A$, which is a convex function of the flow x_{ij},

t_ρ^{ks} : the travel time in path ρ from k to s, $\rho\in\rho^{ks}$ and $(k,s)\in P$,

Then, the user equilibrium (UE) flow problem may be stated as follows (Aashtiani and Magnanti (1981), and Sheffi (1985)):

$$\text{(UE)}\quad (1)\quad x_\rho^{ks}(t_\rho^{ks}-u^{ks})=\circ,\qquad \forall\rho\in\rho^{ks},\forall(k,s)\in P$$

$$(2)\quad (t_\rho^{ks}-u^{ks})\geq\circ,\qquad \forall\rho\in\rho^{ks},\forall(k,s)\in P$$

$$(3)\quad x_\rho^{ks}\geq\circ,\qquad \forall\rho\in\rho^{ks},\forall(k,s)\in P \qquad (19)$$

(4) $u^{ks} \geq \circ, \qquad \forall (k,s) \in P$

(5) $\sum_{\rho \in \rho^{ks}} x_{\rho}^{ks} = d^{ks}, \qquad \forall (k,s) \in P$

where the first three expressions state that if flow on path ρ from k to s is positive then the respective (path) travel time equals that of the shortest path from k to s; otherwise (i.e., if this flow is zero) then the respective travel time is greater than that of the corresponding O/D shortest path; and vice versa. The third expression ensures non-negativity of the path flow, and the fourth the non-negativity of the shortest path travel times. The last expression guarantees the sum of the path flows originating from origin k to destination s to equal the demand for travel from k to s.

This problem may be written equivalently as the following convex programming (CP) problem when link travel times are convex function of the respective flows and the demand is fixed; as assumed:

(UECP) $\text{Min} \quad \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(u)du$

s.t.: (1) $\sum_{\rho \in \rho^{ks}} x_{\rho}^{ks} = d^{ks}, \qquad \forall (k,s) \in P$ (20)

(2) $x_{\rho}^{ks} \geq \circ, \qquad \forall \rho \in \rho^{ks}, \forall (k,s) \in P$

(3) $x_{ij=} \sum_{(k,s)\in P} \sum_{\rho \in \rho^{ks}} x_{\rho}^{ks}.\delta_{ij,\rho}^{ks}, \qquad \forall (i,j) \in A$

where $\delta_{ij,\rho}^{ks}$ equals 1 if (i, j) belongs to path ρ from k to s, otherwise 0. This is the (simplified version of) problem solved by the routine explained in section 3 of the paper, as a traffic assignment problem, in EMME/2 environment.

In a metropolitan area, most of early morning demand constitutes only two trip purposes, namely "work" and "school." (For the study area this totals about 80% of the overall demand during 7 to 9 AM.) Such trips are daily "mandatory" trips, to known destinations. The makers of these trips are, by virtue of the repetition of them, well aware of the network congestion situation, travel times, and other characteristics, and hence consciously choose time of starting the trips, modes of travel, as well as of paths, to minimize their travel times and to reach their destinations on time. These choices are, of course, influenced by uncontrollable parameters (such as trip starting time distribution on demand side, as in our case). Thus, the flow pattern in the network in an hour (and, extending this statement to limits, to say at any time) may be considered to be (basically) in (user) equilibrium (in the sense of UE). Such equilibrium flow may be conceived of running the system for a long time, as in the case of a dynamical system. Figure 6 shows this perception of the system. In the first m days of the time the system is in a transition state, and after this period of learning, equilibrium flow pattern develops. People in urban areas remember this state of equilibrium and take advantage of this information even after a change in season, and adapt themselves to the new situation equilibrium conditions using the information regarding the last such season in the past. This argument shows that daily maximum throughputs are equivalent to running the system for a long time each time the parameter is changed by a small amount. Note that there is a relation between the trajectories and the day clock time, as shown in Figure 6. The right-most figure in Figure 6 shows the trajectories of speed-throughput of the network during equilibrium days which show variations due to change in the parameters of the system (such as time of starting the trip, traffic accidents in the network, etc.). The equilibrium critical (capacity, or maximum) throughputs encircled in this figure are what this paper is concerned about. Note that all q_c^e's are throughputs obtained from a (different) small disturbance of an original set of parameters and are time series of a specific quantity under basically the same conditions (e.g., the throughput of the network during 7-8 AM.) as shown in Figure 7.

Now, let $q_c^{e_1}, q_c^{e_2}, q_c^{e_3}, \ldots, q_c^{e_m}$ be m maximum throughputs of the network observed after a long time of running the system (equilibrium condition). Each q_c^{ei} (i=1,2,...,n) is the result of starting a recurrent system from very early in the morning, and from the same, but infinitesimally disturbed, parameter condition as defined in Section 4.1 for trip starting time and section 4.2 for traffic accidents assuming rapid information propagation (e.g., through certain communication media for this change in parameter).

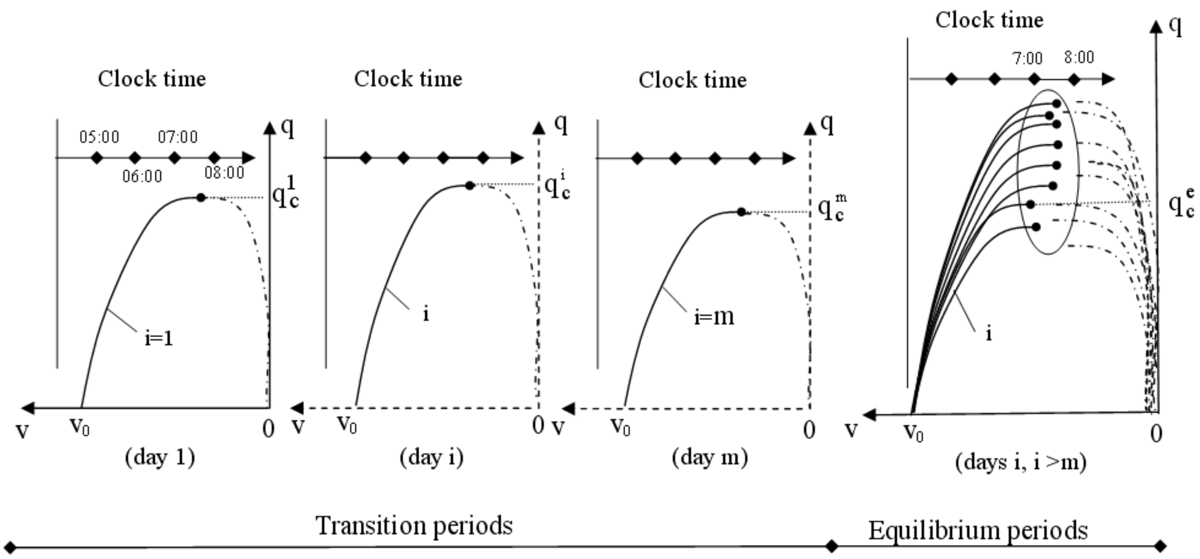


Figure 6- Conceptual speed-throughput trajectories for some transition and equilibrium periods in the morning peak in an urban area.

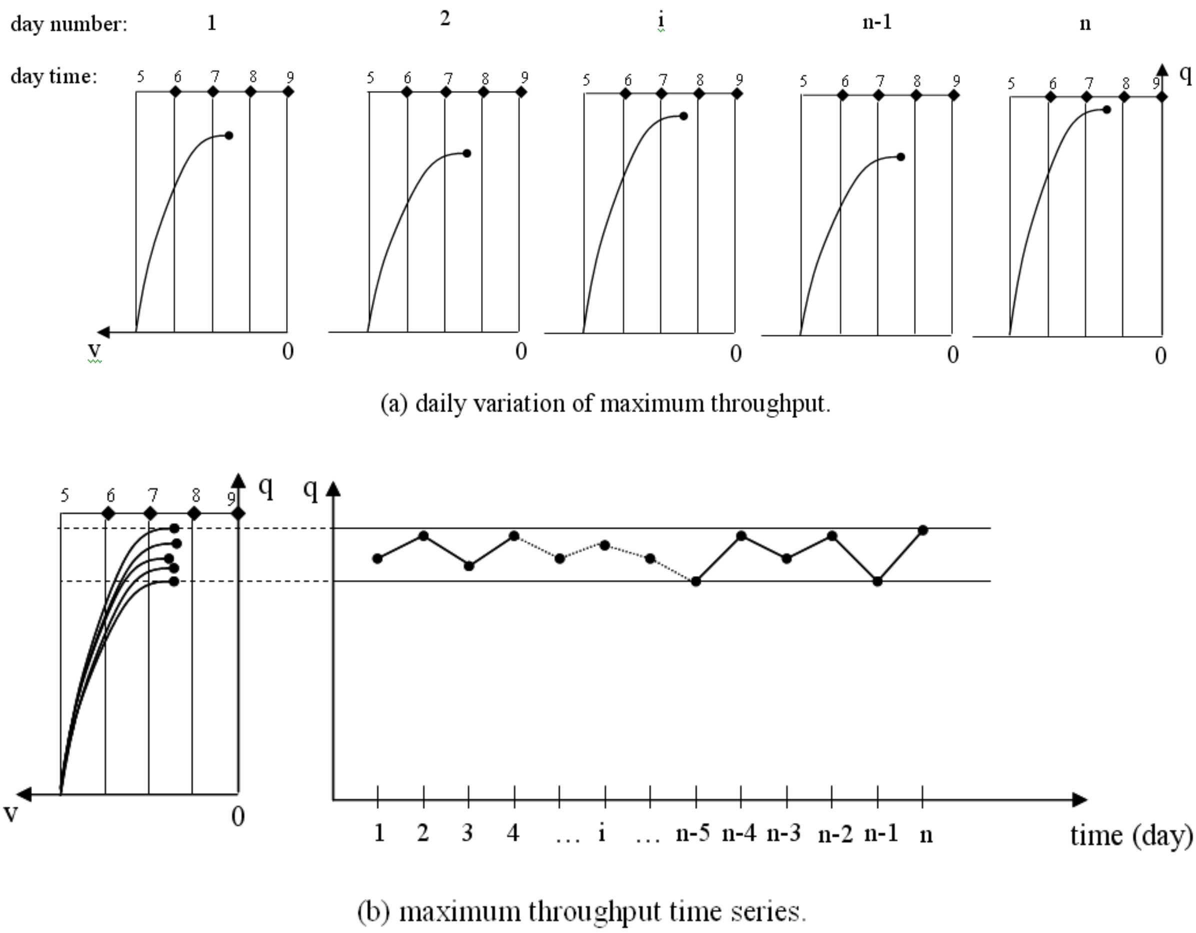


Figure 7- Daily variation of network throughput, and time series of maximum throughput.

Let p_0 denote the initial state of the parameters of the system. The difference between two initially adjacent parameters p_0 and $p_0+\varepsilon$ eventually leads to an equilibrium throughput change from $q_c^e(p_0)$ to $q_c^e(p_0+\varepsilon)$. According to the arguments represented before superscript e represents an equilibrium state resulted from a past learning process so that e is equivalent to the limiting behavior as $n \to \infty$, n being a learning step. Thus, following the usual argument of Lyapunov exponent derivation (see, e.g., Schuster and Just, 2005):

$$(p_0+\varepsilon)-(p_0)=\varepsilon \quad \textit{implies} \quad \left|q_c^e(p_0+\varepsilon)-q_c^e(p_0)\right| = \varepsilon e^{\lambda(p_0)} \tag{21}$$

from which one may write:

$$\frac{\left|q_c^e(p_0+\varepsilon)-q_c^e(p_0)\right|}{\varepsilon} = e^{\lambda(p_0)}$$

or:

$$\lambda(p_0) = \ln\left|\frac{q_c^e(p_0+\varepsilon)-q_c^e(p_0)}{\varepsilon}\right| \tag{22}$$

In order to see an average behavior, and not a single or extreme behavior, of the system one may write Equation 22 for $p^1 = p_0 + \varepsilon^1$, $p^2 = p_0 + \varepsilon^2$, $p^3 = p_0 + \varepsilon^3$, …, $p^m = p_0 + \varepsilon^m$, and then obtain an average λ, where ε^i is the $\varepsilon - \text{deviation}$ from the deterministic state with known original parameter p_0:

$$\lambda(p_0) = \frac{1}{m}\sum_{l=1}^{m} \ln\left|\frac{q_c^e(p_0+\varepsilon^l) - q_c^e(p_0)}{\varepsilon^l}\right| \tag{23}$$

A correct value of λ is when m tends to infinity:

$$\lambda(p_0) = \lim_{m\to\infty}\frac{1}{m}\sum_{l=1}^{m} \ln\left|\frac{q_c^e(p_0+\varepsilon^l) - q_c^e(p_0)}{\varepsilon^l}\right| \tag{24}$$

Now, one may view $\ln|\cdot|$ a as a change in $q_c^e(\cdot)$ as a result of a change from p^l to $p_0 = p^l - \varepsilon^l$, l =1, 2, 3, …, m :

$$\lambda(p_0) = \lim_{m\to\infty}\frac{1}{m}\sum_{l=1}^{m} \ln\left|\frac{q_c^e(p_0) - q_c^e(p^l)}{-\varepsilon^l}\right| \tag{25}$$

In the limits, when $\varepsilon^l\text{'s} \to 0$, assuming that they exist:

$$\begin{aligned}\lambda(p_0) &= \lim_{m\to\infty}\lim_{\varepsilon^l\to 0}\frac{1}{m}\sum_{l=1}^{m} \ln\left|\frac{q_c^e(p^l-\varepsilon) - q_c^e(p^l)}{-\varepsilon^l}\right| \\ &= \lim_{m\to\infty}\frac{1}{m}\sum_{l=1}^{m} \ln\left|\frac{dq_c^e(p^l)}{dp^l}\right| \\ &= \lim_{m\to\infty}\frac{1}{m}\sum_{l=1}^{m} \ln\left|q_c^{e\,\prime}(p^l)\right| \end{aligned} \tag{26}$$

Equation 26 resembles the definition of λ for two trajectories of a dynamical system where $p^l, l = 1,2,...,m$ are the successive states of the system, and hence existing softwares may be used to compute $\lambda(p_0)$ for the network under study.

7. Application of the Theory of Chaos

To see whether the transport network system in this case study become chaotic under a certain condition of demand and supply, 300 observations have been made in the area of (b) in Figure 4. Figure 8 shows aspects of the change in the input parameters, and the throughput of the system in these 300 observations. The reader recalls that two parameters of the system were of concern in the case study in this paper: change in demand due to distribution of arrival times into the network, and closures of link/lane in the network due to traffic accidents. Figure 8a shows the total O/D demand over the average value of this quantity for 300 observations, and Figure 8b shows the total length of the links of the network over the average value of this quantity for 300 observations, as aspects of the variation of the two input parameters. Figure 8c shows the variation of the maximum throughput in these observations in terms of the total O/D demand served satisfactorily over the average value of this quantity for 300 observations. As may be seen in Figure 8, the variation of demand and supply are very little, while that of maximum throughput is appreciable.

Using a software in the environment of MATHLAB (Swope (internet)), the Lyapunov exponent of the series $\left\{ q^{i} = q_c^{ei} / M ; M = \sum_{l=1}^{300} q_c^{el} / 300 ; i = 1,2,...,300 \right\}$ is computed as 0.0236, a positive value, suggesting that the process is chaotic.

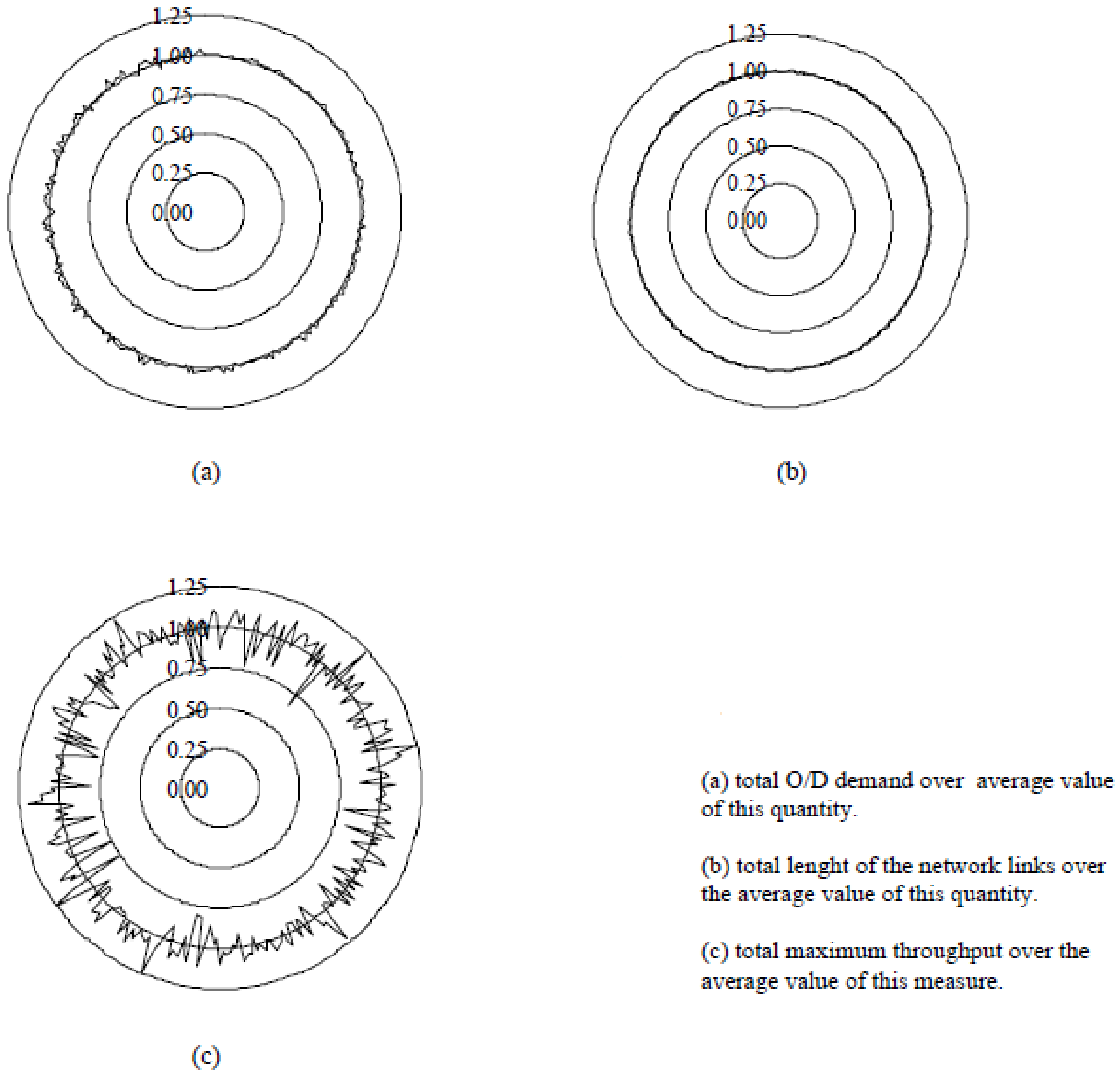


Figure 8- Aspects of input, and throughput, variation in 300 observations of maximum throughputs

8. SUMMARY AND CONCLUSIONS

The paper discussion starts with an observation made by Greenshields in 1934 and several other researchers regarding the speed-flow-density relationships (McShane and Roess (1940)), pointing unstable flow when density of flow reaches a critical point leading to critical speed and flow, at which the system looks “chaotic” (Figures 1 and 2).

Next a model is built for a real network of a metropolitan area to estimate link flows for given demand and network specifications. This model is a deterministic system with many parameters, two of which are selected to be changed by small amounts in order to study the level of the variation of the performance of the network. These two parameters are the demand arrival into the network, which is assumed to be normally distributed with mean (as estimated) and variance about 0.1 of the respective mean value; and the links of the network which may be affected (e.g., closed) by traffic accidents according to certain Poisson processes governing the occurrence of these events.

Network throughput is defined as the sum of O/D trips which are finished satisfactorily. 30 simulation runs have been made to produce network speed-throughput relationship similar to the ones presented by Greenshields and others for a basic freeway section. This is done for the city of Mashhad.

Next, A Lyapunov exponent measure has been devised for a series of equilibrium maximum throughputs similar to that of two initially adjacent trajectories. Using a 300 observation series of maximum throughputs of the study network, it has been shown that the Lyapunov exponent is positive, suggestive of chaotic behavior of the network at the demand level which caused this phenomenon.

An interesting exercise is estimating the population threshold of the city under study, beyond which the network would collapse by chaotic behavior. Suppose D_{cr} is the total demand (in passenger car equivalent per hour, pce/hr) just before the network is driven into chaos. Let ao be the average occupancy of each pce. Then, $ao \cdot D_{cr}$ is the total passengers entering the network at the brink of network break-down. Assuming t vehicular trips per capita during peak hour, an estimate of the population threshold (population capacity) of the city may be estimated as follows:

$$\begin{pmatrix} \text{Population capacity of the city for} \\ \text{the given transportation network} \end{pmatrix} = ao \cdot D_{cr} / (t \cdot f) \qquad (27)$$

where f is the ratio of peak hour O/D demand over daily O/D demand. For the city of Mashhad D_{cr} was about 127,978 pce trips, and $ao \cong 3.65\ person/pce$, $t = 1.6\ trips/day$ and $f = 0.0772$, resulting:

$$\begin{pmatrix} \text{Population capacity of the city for} \\ \text{the given transportation network} \end{pmatrix} = \frac{127{,}978 \times 3.65}{1.6 \times 0.0772} = 3{,}800{,}000\ \ persons$$

The above aggregate computations show that in order to prevent the city network to get into a chaotic situation, one may either observe the population capacity of the city, or expand the network capacity so as to push the maximum throughput to the right in Figure 9. Detail analyses of the problem, using the presented model of city transportation, may reveal where the population capacity should be allocated in the city, or where the network should be expanded to achieve better performance.

Analogous to the case of basic freeway section, network density may be defined and speed-density as well as throughput-density relationships may be derived at network levels, and used for further analyses of the network.

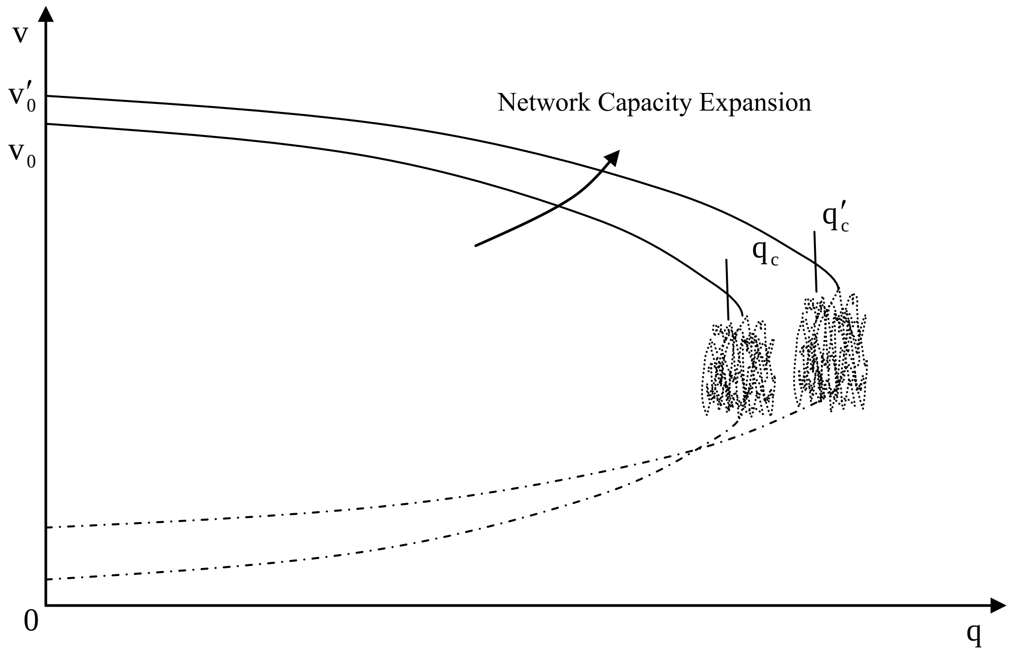


Figure 9- Investment in the network to prevent chaotic behaviour.

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