DEVELOPMENT OF LOCATIONAL EQUILIBRIUM MODELS AND METHOD OF FINDING LOWRY-TYPE EQUILIBRIA

Yevgeny KANTOROVICH
Doctor of Applied Mathematics
Institute of the Master Plan of Moscow,
Moscow - Russia

INTRODUCTION

This paper deals with two types of equilibrium allocation models: equilibrium versions for production-constrained spatial interaction models (Harris and Wilson, 1978; Rijk and Vorst, 1983 ; Roy and Johansson, 1984; Crouchley, 1987) and Lowry-type subsystem interaction models (Wilson, 1974; Batty, 1976; Macgill and Wilson, 1979; Shmulyian, 1979; Kantorovich, 1985, 1986; Kantorovich and Shmulyian, 1986).

In the first type models an equilibrium allocation of subsystem resource over city zones is balanced with flows of subsystem consumers. In particular, there is a balanced allocation of shopping centers, with flows of consumers proportional to chopping capacities. Roy and Johansson (1984) determined and explored the model of this type as a Nash equilibrium model.

Models of the second type are designed for simulation of equilibrium allocations of several interacting urban subsystems. In these models the urban interaction behavior of citizens is described by preference for trips to different subsystems. In (Kantorovich, 1986, 1988), an entropy version of the Lowry model and its generalization are formulated as systems of equations, similar to description of the Nash equilibrium (on dependent sets), which is a solution of the system

$$f^k(\tilde{x}^1,\dots,\tilde{x}^m) = \max_{x^k \in D^k(\tilde{x})} f^k(\tilde{x}^1,\dots,\tilde{x}^{k-1}, x^k, \tilde{x}^{k+1},\dots,\tilde{x}^m)$$

in variables $\tilde{x}^k \in R^{n_k}$, $k = 1 \dots m$ [here $\tilde{x} = (\tilde{x}^1,\dots,\tilde{x}^m)$]. The variable $\tilde{x}^k$ is usually called a strategy of the k-th player, and the function f^k is called a payoff function.

The problems of searching subsystem equilibria in the above-mentioned models can be reduced to the problems of searching fixed points of the correspondent maps. In this paper the main attention is paid to problems of description, searching and studying equilibria in the equilibrium allocation models with zonal-capacity constraints. For this purpose we use the map

$$\varphi^V(t) = \underset{x \in V}{\operatorname{argmax}}\ H(x,t) = \underset{x \in V}{\operatorname{argmax}} \sum_i x_i \ln \frac{t_i}{x_i}, \quad t \in R^n_+ ,$$

onto a set

$$V = \{x \mid q_i^{min} \le x_i \le q_i^{max}, \ i = 1...n, \ \sum_i x_i = Q\} \subset R^n_+$$

[in (Kantorovich, 1988) this map is called an entropy projection onto V]. We also use Brouwer fixed point theorem and the principle of compressing map for finding and studying equilibria. The structure of the paper is as follows.

In Section 1, a formal description of equilibria of interacting subsystems is presented, and the main results of studying map φ^V are given.

Section 2 is devoted to equilibria in models of urban subsystem allocation. An entropy version of Lowry service allocation submodel is studied from this viewpoint. The model proposed by Harris and Wilson (1978) is enriched with zonal-capacity constraints for allocation. For this modification, uniqueness of an equilibrium, and convergence of the computational algorithm are studied.

Section 3 presents an entropy version of the Lowry model formulated as an equilibrium of interacting subsystems.

1. DESCRIPTION OF EQUILIBRIA

In (Rosen, 1965), the Nash equilibrium with dependent sets of strategies is determined using a convex, closed and bounded set D, containing feasible states $\tilde{x}$, and using sets

$$D^k(\tilde{x}) = \{ x^k \mid (\tilde{x}^1, ..., \tilde{x}^{k-1}, x^k, \tilde{x}^{k+1}, ..., \tilde{x}^m) \in D \}$$

which are projections of 'sections' of set D to R^{n_k}.

In (Makarov and Rubinov, 1973), a more general definition of Nash equilibrium state is suggested. Namely, the set D is described implicitly by maps D^k and can be non-convex.

Let X^k be the set of strategies of the k-th player, $X = X^1 \times X^2 \times ... \times X^m$ and let $D^k : x \longrightarrow D^k(x) \subset X^k$ be map taking points of X to subsets of X^k, k = 1...m.

A point $\tilde{x} \in X$ is called a feasible state of the game, if $\tilde{x}^k \in D^k(\tilde{x})$ for any k. So, the set of feasible states

D is determined by maps D^k, but not vice versa. This description of an equilibrium state supposes payoff functions to be maximized over sets $D^k(\tilde{x})$. In general, such a set does not coincide with a projection of the corresponding section (through $\tilde{x}$) of the set of feasible states D onto R^{n_k}. Existence of the Nash equilibrium was proved by Makarov and Rubinov (1973) with the following assumptions: i) for any k = 1...m, X^k is convex and compact in the finite-dimensional space R^{n_k}, and D^k is continuous in Hausdorff metrics; ii) f^k is continuous in X and concave with respect to x^k ; iii) for any $x \in X$, the set $D^k(x)$ is convex. Similar assumptions (the last one holds by the definition) are used by Rosen (1965) for proving the existence theorem.

Let $x^k = (x_1^k, \ldots, x_n^k) \in R^{n_k}$, $x = (x^1, \ldots, x^m)$.

We define a generalization of Nash equilibrium formally as a solution of the system (in $\tilde{x}^1, \ldots, \tilde{x}^m$)

$$f^k(\tilde{x}^k, \tilde{x}) = \max_{x^k \in D^k(\tilde{x})} f^k(x^k, \tilde{x}), \quad k = 1...m, \qquad (1)$$

in which the sets $D^k(x)$ are defined in one of the ways we have considered. Applying this to interaction of urban subsystems, we will assume that

$$f^k(x^k, \tilde{x}) = H(x^k, t^k(\tilde{x})) = \sum_{i=1}^{n_k} x_i^k \ln \frac{t_i^k(\tilde{x})}{x_i^k}, \qquad (2)$$

$t^k : X \to R_+^{n_k}$ being a map, $t_i^k(\tilde{x})$ being coordinates of $t^k(\tilde{x})$.

The system (1), (2) includes some entropy modifications of the Lowry model and is regarded with respect to flows of population or with respect to capacities of subsystems in city zones (Kantorovich, 1986, 1988).

In models of the second type, the map t^k reflects behavior of population in trips and determines preferences or probabilities of allocation for k-th subsystem in city zones. The difference between the system (1) and the definition of Nash equilibrium consists in dependence of the function f^k on equilibrium value $\tilde{x}^k$.

It should be noted, that with values of all the 'strategies' fixed except the k-th one, solving of the k-th equation of the system (1) consists not in maximization of the payoff function, but in finding the fixed point of map

$$\tilde{x}^k \longrightarrow \underset{x^k \in D^k(\tilde{x})}{\operatorname{argmax}} \; f^k(x^k, \tilde{x}^1, \ldots, \tilde{x}^k, \ldots, \tilde{x}^m)$$

By doubling the dimension of the model (1), we can 'reduce' it to Nash equilibrium, i.e. the set of solutions of (1) coincides with the set of solutions of some system describing Nash equilibrium.

It is essential that the model (1), (2) formalizes interaction of urban subsystems by using not only mutual dependence of constraints but also dependence of entropy functions to be maximized.

Taking into account further use of capacity constraints in the urban models of equilibrium, the system (1) will be studied with maximizing problems of the following type:

$$\underset{x}{\text{maximize}} \quad H(x,t) = \sum_i x_i \ln \frac{t_i}{x_i} \tag{3}$$

subject to

$$\begin{aligned} &q_i^{min} \le x_i \le q_i^{max}, \quad i = 1\ldots n, \\ &\sum_i x_i = Q. \end{aligned} \tag{4}$$

In (Kantorovich, 1985, 1986) the map

$$\varphi^V(t) = \underset{x \in V}{\operatorname{argmax}} \; H(x,t) = \underset{x \in V}{\operatorname{argmax}} \sum_i x_i \ln \frac{t_i}{x_i} \tag{5}$$

(here V is determined by constraints (4); $t_i > 0$, $i = 1\ldots n$) and more general map

$$h(x) = \underset{y \in D(x)}{\operatorname{argmax}} \; H(y, t(x)) = \varphi^{D(x)}[t(x)]$$

are studied.

It should be noted that the map φ^V is not always differentiable

The method and the main results of studying maps φ^V and h are given in (Kantorovich, 1988), while the detailed proofs are in (Kantorovich, 1986). In these papers the idea of Bregman's balancing method [or consecutive D-projections (Bregman, 1967)] known from convex programming is employed. Let us present some of these results used further for equilibrium models.

Consider an n-dimensional real space R_1^n with the following norm, metrics and partial order:

$$\| x \| = \sum_{i=1}^{n} |x_i| , \quad \rho(x^1, x^2) = \sum_{i=1}^{n} |x_i^1 - x_i^2| ,$$

$$x^1 \le x^2 \text{ , if and only if } x_i^1 \le x_i^2 \text{ for } i = 1...n.$$

The map $\varphi^V(t)$, with

$$V = \{ x | q^{min} \le x \le q^{max} , \| x \| = Q \},$$

will be considered in the simplex

$$S = \{ t | t > 0, \| t \| = Q \}$$

Let $\varphi_i^V(t)$ be the i-th coordinate of $\varphi^V(t)$.

Theorem 1. The map $\varphi^V(t)$ is continuous in S and there exists a real continuous function $\psi^V(t)$, such that

$$\varphi_i^V(t) = \max \{ q_i^{min}, \min \{ q_i^{max}, \psi^V(t)t_i \}\}$$

Theorem 2. If $t^\circ \in S$, then

$$\rho[\varphi^V(t^\circ), \varphi^V(t)] \le \psi^V(t^\circ)\rho(t^\circ, t)$$

for any $t \in S$.

Theorem 3. Let $\tilde{\varphi}^V$ be the restriction of the map φ^V to the set $\tilde{S} \subset \{ t | t \ge r > 0, t \in S \}$, then $\psi^V(r)$ is a Lipschitz constant of the map $\tilde{\varphi}^V$.

Theorem 4. If $q_i^{max} = +\infty$, i=1...n, then $\psi^V(t) \le 1$ for any $t \in S$, and φ^V is a non-stretching map.

We underline that Theorem 2 yielding a Lipschitz constant in the point t° is of a global nature, because the 'free point' t is arbitrary in S. It is worth be notice that the Lipschitz constant $\psi^V(t^\circ)$ is non-decreasing in any metrics generated by a norm in a real n-dimensional linear space. Theorem 4 cannot be 'improved', since φ^V is identical on V. Thus, the selected metrics is 'suitable' for applying compressing properties of the map φ^V.

2. EQUILIBRIUM ALLOCATION OF AN URBAN SUBSYSTEM

In this section, an equilibrium allocation of objects is understood as an allocation balanced with flows of clients to these objects.

Let us consider two models of this type. Let $q = (q_1, \ldots, q_n)$ is a distribution of service among n city zones, p and b are constant vectors corresponding to distribution of population and of basic employment, respectively, A_i are matrices (with positive elements) of preference for trips to service places in various zones of the city.

Let us consider a natural entropy version of the Lowry submodel of service allocation

$$q = \varphi^V[A(q)] \quad (6)$$

where the vector of allocation 'potentials'

$$A(q) = \alpha_1 A_1 q + \alpha_2 A_2 p + \alpha_3 A_3 b$$

is obtained by summing flows of clients of various categories to city zones. The map φ^V with

$$V = \{ q \mid 0 \le q^{min} \le q \le q^{max}, \ \|q\| = Q \},$$

the norm $\|\cdot\|$, the partial order $\le$ and the metrics ρ of the space R^n_1 henceforth are the same as in Section 1.

The equation (6) concerning q can be regarded as a model for service allocation (with fixed distribution of population $\{p_i\}$). A solution $q^* = (q_1^*, \ldots, q_n^*)$ of the equation (6) will be called an equilibrium allocation, or an equilibrium.

Existence of an equilibrium allocation q^* follows from Theorem 1 and from Brouwer fixed point theorem. The linear operators given by the stochastic matrices A_i are non-stretching in the metrics ρ. From Theorem 2, we obtain sufficient conditions for uniqueness of equilibrium q^*:

$$L(q^*) = \alpha_1 \psi^V[A(q^*)] < 1 . \quad (7)$$

The map

$$f(q) = \varphi^V[A(q)]$$

takes the compact V into itself, and its Lipschitz constant $L(q^*)$ in the point q^* is of a 'global nature'. Thus the condition (7) guarantees convergence of the iterative process

$$q^{m+1} = f(q^m) \quad (8)$$

(starting to q^* from any initial state $q^\circ \in V$).

Let $r = \alpha_1 A_1 q^{min} + \alpha_2 A_2 p + \alpha_3 A_3 b$. From Theorem 3 we obtain a Lipschitz constant and a sufficient condition for compressibility of the map f:

$K = \alpha_1 \psi^V(r) < 1$.

In this case, an equilibrium allocation is unique and the process (8) converges to it from any initial state. We note that if $q_i^{max} = +\infty$ (i.e. if zonal-capacity constraints are of the form $q_i \geq q_i^{min}$, $i = 1...n$), then it is easy to prove compressibility of the map f . In a general case, the above mentioned sufficient conditions are convenient to apply, and verifying them does not require long computations.

Let us consider the equilibrium model (Harris and Wilson, 1978), In an equivalent form, an equilibrium allocation q^* can be defined as one satisfying the equation

$q = kA(q)b$

in which the column-vector b consists of volumes of sources in the zones, and elements of the matrix A(q) are:

$$a_{ji}(q) = g(q_j)\nu(c_{ij}) / \sum_k g(q_k)\nu(c_{ik}) .$$

Here the coefficient $k = \|q\|/\|b\|$ is for converting flows of consumers into capacities (in this case, k is constant for all zones), g is an attractiveness function, c_{ij} is the travel cost from i to j, ν converts travel cost into a representation of impedance. $\sum_j a_{ji}(q) = 1$ holds for any q, and elements $a_{ji}(q)$, $j=1...n$, are relative amounts of clients, moving from the zone i to various zones of the city. Multiplication of matrix A(q) by column-vector b is standard, and the flow of clients to the j-th zone is

$$D_j(q) = \sum_i a_{ji}(q)b_i$$

Let us formulate the following development of the given model:

$$q = \varphi^V[kA(q)b] \qquad (9)$$

with $V = \{q \mid 0 < q^{min} \leq q \leq q^{max}, \|q\| = Q\}$

If $q^{min} \geq 0$, it is possible to prolong the map $f(q) = \varphi^V[kA(q)b]$ continuously to the frontier of the set V. Introduction of capacity constraints into the model allows to use additional criteria of allocation besides balancing condition of flows and to avoid balanced but undesirable allocations (such as an allocation of the total resource in a single zone).

Let

$$\tilde{D}(q) = (\tilde{D}_1(q), \ldots, \tilde{D}_n(q)) = kD(q) = kA(q)b ,$$

$$\|\nabla\tilde{D}(q)\| = \max_j \sum_i \left| \frac{\partial\tilde{D}_i(q)}{\partial q_j} \right| ,$$

i.e. the norm of Jacobi matrix $\nabla\tilde{D}(q)$ corresponds to the norm of the space R_1^n.

The map f(q) taking the compact V into itself is not differentiable everywhere in V. However, from Theorem 2, it follows that if q^* is a solution of the equation (9), then the condition

$$L(q^*) = \psi^V[\tilde{D}(q^*)]\|\nabla\tilde{D}(q^*)\| < 1 \qquad (10)$$

guarantees local convergence of the iterative process

$q^{m+1} = f(q^m)$,

i.e. there exists a neighborhood $U \ni q^*$ such that the sequence $\{q^m\}$ starting from $q^\circ \in U \cap V$, converges to q^*.

Moreover, if $L(q) \le \beta < 1$ holds for any $q \in V$, then the map f(q) is compressing.

Using the results of Harris and Wilson, (1978) it is not difficult to obtain that

$$\|\nabla D(q)\| = 2 \max_j \frac{\partial D_j(q)}{\partial q_j} \le \max_j \frac{\|b\| \; g'(q_j)}{2 \; g(q_j)}, \qquad (11)$$

As an example, we consider the attractiveness function $g(r) = r^\alpha$, $r \in R$. For the case

$V = \{q | \; q_j \ge q_j^{min}, \; \|q\| = Q\}$

using (10), (11) and Theorem 4 we can show that for any equilibrium q^*, the inequality $\alpha < 0.5$ guarantees local convergence of $\{q^m\}$, and the inequality $K = 0.5\alpha Q \max_j \{1/q_j^{min}\} < 1$ guarantees compressibility of map f in V, i.e. the uniqueness of an equilibrium and convergence of iterations $q^{m+1} = f(q^m)$.

3. EQUILIBRIA OF INTERACTING SUBSYSTEMS

In this Section, using the entropy projection operator, we will construct the versions of equilibrium model of interacting subsystems, which is 'very close' to the classical Lowry model. Different types of constraints are used here to describe different priorities for allocation of interacting subsystems.

Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ be distribution of employed population and of people employed in service among n zones. For the sake of simplicity consider the service as aggregated in one type. The distribution of basic employment $b = (b_1, \ldots, b_n)$ is supposed to be fixed. We determine the equilibrium allocation of subsystems p^*, q^* as the solution of the non-linear system (in p, q):

$$p = \varphi^{V^1}[B(q)], \qquad q = \varphi^{V^2}[A(p, q)] \tag{12}$$

in which $B(q)=B_1 q + B_2 b$ and $A(p,q) = \alpha_1 A_1 q + \alpha_2 A_2 p + \alpha_3 A_3 b$ are maps yielding 'potentials' of subsystems allocation. The input for the model is represented by stochastic matrices B_i, A_i, of preferences for trips to residence and to service respectively, by coefficients α_i expressing relative amounts of clients of different categories, by the vector b, by resources of subsystems $P = \|p\|$, $Q = \|q\|$ and by vectors q^{min}, q^{max}, p^{min}, p^{max}.

We consider three types of zonal-capacity constraints.

In the first type, priorities for allocation suppose to be define (for example by planning authority) using minimum and maximum zonal-capacity constraints:

$$V^1 = \{p \mid p^{min} \le p \le p^{max}, \ \|p\| = P\}$$

$$V^2 = \{q \mid q^{min} \le q \le q^{max}, \ \|q\| = Q\}$$

Competition for land use can be introduced to the model by the constraint $\beta_1 p + \beta_2 q \le d$, i.e. by the following feasible sets:

$$V^1(q) = \{p \mid p^{min} \le p \le (d - \beta_2 q)/\beta_1, \ \|p\| = P\}$$

$$V^2(p) = \{q \mid q^{min} \le q \le (d - \beta_1 p)/\beta_2, \ \|q\| = Q\}$$

Competition of subsystems for territory accepted in the Lowry model may be represented by dependent constraints

$$V^1(q) = \{p \mid p^{min} \le p \le p^{max} - \beta q,\ \|p\| = P\}$$

$$V^2 = \{q \mid q^{min} \le q \le q^{max},\ \|q\| = Q\}$$

This dependence of constraints formalizes classical priority of service allocation, since in the equilibrium distribution p^*, q^*, the point p^* maximizes entropy of the population subsystem over the 'set of states' depending on the territory free from service.

Note, finally, that the system (12) does not coincide with the description of the Nash equilibrium, but this system is a particular case of the system (1), (2). The first couple of constraints describe the independent sets for allocation as in the classical Nash equilibrium, the second one describe the dependent sets as in (Rosen, 1964), and the third one describe the dependent sets as in (Makarov and Rubinov, 1973). Using the theorems from Section 1 it is possible to obtain conditions for the uniqueness of an equilibrium of interacting subsystems and for the convergence of the computational algorithm.

4. CONCLUSION

Let us consider some aspects of equilibrium models we have discussed.

Model (6) is an entropy analog of the Lowry service allocation submodel. Service allocation influences to flows of clients so far as people employed in the service are also clients of service, and so this model is 'close' to the equilibrium version of production-constrained spatial-interaction model (Harris and Wilson, 1978). Involving this category of clients in the model leads to the problem of searching a fixed point of the correspondent map. It should be noted that sufficient conditions of compressibility of this map are usually satisfied in applications, and this is mainly explained by the fortunate choice of space metrization.

The equilibrium allocation model from (Harris and Wilson, 1978) reflects attractiveness of zonal capacities and is much more adequate to interaction of citizens and the functional subsystem. Fixed point theorems can also be used to study this model [e.g. (Rijk and Vorst, 1983)]. However involving attractiveness functions [especially, those dealing with spatial competition and agglomeration (Fotheringham, 1985)] leads to additional non-linearities in equilibrium models of spatial interaction and makes their study more complicated. In fact, a version of this model (9) has been considered in our paper for rather particular case.

Limits for applying the correspondent compressibility condition are narrower then for a similar condition in the model (6). In practice, this condition can be satisfied for rather small α in problems of equilibrium allocation of several large centers.

The concept of equilibrium of the interacting urban subsystems can be formalized by a rather general model (1), (2); each equation of the system (1) describes an equilibrium allocation of a particular subsystem. Models from Section 2 can be described by one equation of this type, and the Lowry-type model (Section 3) we describe by two equations. Note that models of the type (1), (2) are simulational, and thus the problem of description of their sets of equilibria is rather important. Theorems 1 - 4 from Section 1 allow to study equilibrium Lowry-type models with locational-capacity constraints. Investigation of similar models with attractiveness functions seems to be interesting but rather difficult.

BIBLIOGRAPHY

Batty M. (1976) Urban modelling: algorithms, calibrations, predictions. Cambridge University Press, Cambridge, England

Bregman L. (1967) The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. U.S.S.R., Computational Math. Mathematical Phys., 7, 200-217.

Crouchley R. (1987) An examination of the equivalence of three alternative mechanisms for establishing the equilibrium solutions of the production-constrained spatial interaction model. Environment and Planning A, 19, 861-874.

Fotheringham A.S. (1985) Spatial competition and agglomeration in urban modelling. Environment and Planning A, 17, 213-230

Harris B. and Wilson A.G. (1978) Equilibrium values and dynamics of attractiveness terms in production-constrained spatial interaction models. Environment and Planning A, 10, 371-388.

Kantorovich Ye.G. (1985) Sufficient conditions of convergence for an iterative procedure of an aggregated allocation model of urban subsystems. Metody Issledovanija Slozhnykh System, 58-64, Institute for Systems Studies, Moscow (in Russian).

Kantorovich Ye.G. (1986) Application of a projection method for study of equilibria in the Lowry model. Sbornik Trudov,9, 74-89, Institute for Systems Studies, Moscow (in Russian).

Kantorovich Ye.G. (1988) Consecutive projection method in the analysis of spatial interaction urban models. Sistemi Urbani, 1988, 2/3, 131-149.

Kantorovich Ye.G. and Shmulyian B.L. (1986) Study of stability for the equilibrium state of an aggregated entropy model describing an allocation of urban subsystems. Avtomatika i Telemekhanika, 11, 71-81 (in Russian).

Makarov V.L. and Rubinov A.M. (1973) Mathematical theory of economical dynamics and equilibria. Nauka, Moscow (in Russian)

Macgill S.M. and Wilson A.G. (1979) Equivalences and similarities between some alternative urban and regional models. Sistemi Urbani, 1, 9-40.

Rijk F.J.A. and Vorst A.C.F. (1983) On the uniqueness and existence of equilibrium points in a urban retail model. Environment and Planning A, 15, 475-482.

Rosen J.B. (1965) Existence and uniqueness of equilibrium points for concave N-person games. Econometrica, 33, 3, 520-534.

Roy J. and Johansson B. (1984) On planning and forecasting the location of retail and service activity. Regional Science and Urban Economics, 14, 433-452.

Shmulyian B.L. (1979) Entropy model of urban system. Avtomatika i Telemekhanika, 10, 70-83 (in Russian).

Wilson A.G. (1974) Urban and regional models in geography and planning. John Wiley and Sons, London.