MODELLING TRANSPORTATION TARGETS IN SPATIAL MARKETS VIA VARIATIONAL INEQUALITIES

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INTRODUCTION

Societal concern with national energy policy, water and air pollution, global warming, the transportation of hazardous materials, and urban congestion is bringing the study of negative externalities to the fore. The development of methodologies for the analysis of negative economic externalities is, hence, a topic which merits increasing attention.

Policy interventions such as taxes in the case of economic systems and tolls in transportation networks have been utilized for many years in order to ensure that user - optimizing behavior approaches what is, in some sense, optimal for the system. In this paper we focus on the problem of modelling transportation goal targets in perfectly competitive markets. The framework which we develop is a general one in that no optimization formulation of the resulting equilibrium conditions needs to exist. The concepts introduced here can, hence, be applied in different economic settings.

For example, the motivation for such a problem is illustrated via the following sample scenario. Consider a number of spatially separated regions involved in the production/consumption of a commodity. Suppose that a decision-maker is concerned with possible overshipment between regions, resulting in environmental damage, depletion of natural resources, or other negative economic externalities. He then formulates target transportation goals with assessed penalties for failure to comply. The assessed penalties may be different for overshipment and undershipment. The resulting equilibrium production, consumption, shipment, and over or undershipments are then to be determined.

In this paper we utilize the theory of variational inequalities to formulate, analyze, and solve such a market model. We also interpret the penalties associated with over/undershipments in terms of taxes/subsidies. In the absence of targets and associated penalties, the model reduces to the spatial market model studied in Nagurney (1987). Variational inequality theory has been applied to model policy interventions in the form of price supports on the production and consumption sides; in this case the markets need no longer clear and the state is one of disequilibrium (Nagurney and Zhao (1991)). In this paper, in contrast, we introduce policy interventions on the transportation (or trade) links in the form of goal targets.

1. THE MODEL

In this Section we introduce the perfectly competitive model with target transportation goals. That is, we consider goals imposed on the levels of commodity shipments between regions which are to be met. Deviations from transportation targets in the case of overshipment or undershipment are subjected to unit penalties.

We first define some notation. Let $i = 1, \ldots, m$ denote the supply regions and let $j = 1, \ldots, n$ denote the demand regions. Let $s = [s_i]$ denote the column vector of supplies, $d = [d_j]$ the column vector of demands, and $Q = [Q_{ij}]$ the matrix of nonnegative commodity shipments from the supply regions to the demand regions. The matrix of commodity transportation targets is denoted by $Q^{\#} = \left[Q^{\#}_{ij}\right]$, with the column vectors of nonnegative overshipments and undershipments denoted, respectively, by $\Delta^+ = \left[\Delta^+_{ij}\right]$, and $\Delta^- = \left[\Delta^-_{ij}\right]$.

The conservation of flow equations are:

$$s_i = \sum_{j=1}^{n} Q_{ij}, \quad i = 1, \ldots, m \tag{1}$$

$$d_j = \sum_{i=1}^{m} Q_{ij}, \quad j = 1, \ldots, n \tag{2}$$

$$Q_{ij} - \Delta^+_{ij} + \Delta^-_{ij} = Q^{\#}_{ij}, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n. \tag{3}$$

Equations (1) and (2) state, respectively, that the supply of a commodity at a region must be equal to the commodity shipments out of the region, and the demand for a commodity at a region must be equal to the commodity shipments into the region. Equation (3) reflects that the transportation goal between the supply region i and demand region j is equal to the commodity shipment between the pair of regions minus the possible overshipment above the goal plus the possible undershipment below the goal. We then denote the closed convex subset of $R^{m+3mn+n}$ consisting of $(s, Q, d, \Delta^+, \Delta^-)$ such that (1), (2), and (3) hold by K.

Let π_i denote the supply price of the commodity at region i and let ρ_j denote the demand price of the commodity at region j. We group the supply prices into a row vector $\pi \in R^m$ and the demand prices into a row vector $\rho \in R^n$. The unit transportation cost associated with shipping the commodity between regions i and j is denoted by c_{ij} and the row vector of transportation costs by $c \in R^{mn}$.

Let M_{ij} denote the unit penalty to be assessed for an overshipment of the commodity between regions i and j and let N_{ij} denote the unit penalty to be assessed for an undershipment of the commodity between the pair of regions. We group the overshipment penalties into a row vector $M \in R^{mn}$ and the undershipment penalties into a row vector $N \in R^{mn}$.

We assume, as in general spatial market models, that the supply price at a region may depend upon the entire vector of supplies and, similarly, the demand price at a region may depend upon the entire vector of demands, that is,

$$\pi = \pi(s) \tag{4}$$

$$\rho = \rho(d) \tag{5}$$

where π and ρ are known smooth functions.

The transportation cost associated with shipping the commodity between a pair of regions may, in general, depend upon the entire commodity shipment pattern, that is,

$$c = c(Q) \tag{6}$$

where c is a known smooth function.

In the absence of target transportation goals and associated penalties of overshipment and undershipment, the well-known spatial price equilibrium conditions, following Samuelson (1952), and Takayama and Judge (1971), are given by: For each pair of regions (i, j), with the equilibrium pattern (s, Q, d) satisfying constraints (1) and (2), we must have that

$$(\pi_i(s) + c_{ij}(Q) - \rho_j(d)) \times Q_{ij} = 0 \tag{7}$$

$$\pi_i(s) + c_{ij}(Q) - \rho_j(d) \geq 0, \quad Q_{ij} \geq 0. \tag{8}$$

The well-known conditions (7) and (8) simply state that a commodity will be shipped between a pair of regions, provided that the supply price at the producing region plus the cost of transportation is equal to the demand price at the consuming region. There will be no shipment if the supply price plus the transportation cost exceeds the demand price. As has been shown in Florian and Los (1982) (see, also, Friesz, et al.), the variational inequality formulation is then given by:

Determine (s, Q, d) satisfying (1) and (2) such that:

$$\pi(s)\cdot(s'-s)+c(Q)\cdot(Q'-Q)-\rho(d)\cdot(d'-d) \geq 0, \ \forall (s', Q', d') \text{ satisfying (1) and (2).} \tag{9}$$

In the presence of target goals, the governing equilibrium conditions now take on the following expanded form: For all pairs of regions (i, j), with the equilibrium pattern $(s, Q, d, \Delta^+, \Delta^-)$ satisfying constraints (1), (2), and (3), we must have that

$$(\pi_i(s) + c_{ij}(Q) - \rho_j(d) + t_{ij}) \times Q_{ij} = 0 \tag{10}$$

$$\pi_i(s) + c_{ij}(Q) - \rho_j(d) + t_{ij} \geq 0, \quad Q_{ij} \geq 0 \tag{11}$$

and

$$t_{ij}\begin{cases} = M_{ij}, & \text{if} \quad Q_{ij} > Q_{ij}^{\#} \\ \in [-N_{ij}, M_{ij}], & \text{if} \quad Q_{ij} = Q_{ij}^{\#} \\ = -N_{ij}, & \text{if} \quad Q_{ij} < Q_{ij}^{\#} \end{cases} \tag{12}$$

or, equivalently,

$$(t_{ij} - M_{ij})\Delta_{ij}^{+} = (t_{ij} + N_{ij})\Delta_{ij}^{-} = 0 \tag{13a}$$

$$-N_{ij} \le t_{ij} \le M_{ij}. \tag{13b}$$

We now interpret the above equilibrium conditions. The value of t_{ij} is interpreted as an excise tax levied on the commodity shipments between regions when it is positive or $-t_{ij}$ is interpreted as unit subsidy paid to shippers between regions i and j when t_{ij} is negative.

In the case of overshipment above the goal, the excise tax levied on the shipments will be equal to the unit penalty for overshipment. On the other hand, if there results an undershipment below the target goal, then a unit subsidy of the same amount as the unit penalty for undershipment should be paid. Finally, if the transportation target goals can be attained precisely, then there may be a tax or a subsidy. The excise tax cannot exceed the unit penalty of overshipment; the subsidy cannot exceed the penalty for undershipment.

Note that condition (13a) implies that for a given pair of regions (i, j) there can only be either an overshipment or an undershipment.

We now derive the variational inequality formulation of the equilibrium conditions governing the above model, that is, we establish the following:

Theorem 1: A pattern of supplies, commodity shipments, demands, and overshipment and undershipment levels $(s, Q, d, \Delta^{+}, \Delta^{-})$ in K satisfies the economic equilibrium conditions (10), (11), (12), (13a), and (13b) in the presence of transportation target goals and associated penalties if and only if it satisfies the variational inequality:

$$\pi(s)\cdot(s'-s)+c(Q)\cdot(Q'-Q)-\rho(d)\cdot(d'-d)+M\cdot(\Delta^{+'}-\Delta^{+})+N\cdot(\Delta^{-'}-\Delta^{-}) \ge 0 \tag{14}$$

$$\forall (s', Q', d', \Delta^{+'}, \Delta^{-'}) \in K.$$

Proof: We first establish necessity. From (10), we obtain that for each (i, j) pair

$$(\pi_i(s) + c_{ij}(Q) - \rho_j(d)) \times Q_{ij} = -t_{ij}Q_{ij},$$

whereas from (13a), we conclude that $t_{ij}\Delta_{ij}^{+}=M_{ij}\Delta_{ij}^{+}$, $-t_{ij}\Delta_{ij}^{-}=N_{ij}\Delta_{ij}^{-}$. Therefore,

$$\sum_{ij}(\pi_i(s) + c_{ij}(Q) - \rho_j(d))Q_{ij} + \sum_{ij} M_{ij}\Delta_{ij}^{+} + \sum_{ij} N_{ij}\Delta_{ij}^{-}$$

$$= \sum_{ij} -t_{ij}Q_{ij} + \sum_{ij} t_{ij}\Delta_{ij}^{+} + \sum_{ij} -t_{ij}\Delta_{ij}^{-} = -\sum_{ij} t_{ij}Q_{ij}^{\#}. \tag{15}$$

On the other hand, by applying (13b), we have that

$$\sum_{ij}(\pi_i(s) + c_{ij}(Q) - \rho_j(d))Q'_{ij} + M_{ij}\Delta_{ij}^{+'} + N_{ij}\Delta_{ij}^{-'}$$

$$\geq \sum_{ij} -t_{ij}Q'_{ij} + t_{ij}\Delta_{ij}^{+'} - t_{ij}\Delta_{ij}^{-'} = -\sum_{ij} t_{ij}Q_{ij}^{\#}. \tag{16}$$

Hence, (14) must hold.

We now turn to sufficiency. Clearly Q, Δ^+, Δ^- is a solution to the following linear programming problem:

$$\text{Minimize} \quad \sum_{ij}(\pi_i(s) + c_{ij}(Q) - \rho_j(d))Q'_{ij} + \sum_{ij} M_{ij}\Delta_{ij}^{+'} + \sum_{ij} N_{ij}\Delta_{ij}^{-} \tag{17}$$

subject to $Q', \Delta^{+'}, \Delta^{-'} \geq 0$, and satisfying (3).

From the Kuhn-Tucker conditions to the above problem we recover the equilibrium conditions (10), (11), (12), and (13), where t_{ij} is the dual variable associated with constraint (3).

We emphasize that in the case where the Jacobian matrices of the supply price, demand price, and transportation cost functions are symmetric, the above equilibrium conditions can be reformulated as an optimization problem, as was observed by Beckmann, McGuire, and Winsten (1956) in the framework of the Wardropian traffic equilibrium conditions, and by Samuelson (1952) and Takayama and Judge (1971) in the case of spatial price equilibrium conditions. Indeed, in this special case, the above equilibrium conditions are the Kuhn-Tucker conditions of the following problem:

$$\text{Minimize} \quad \sum_{i=1}^{m} \int_0^{\sum_{j=1}^{n} Q_{ij}} \pi_i(x)dx + \sum_{i=1}^{m}\sum_{j=1}^{n} \int_0^{Q_{ij}} c_{ij}(y)dy - \sum_{j=1}^{n} \int_0^{\sum_{i=1}^{m} Q_{ij}} \rho_j(z)dz$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\Delta_{ij}^{+} + \sum_{i=1}^{m}\sum_{j=1}^{n} N_{ij}\Delta_{ij}^{+} \tag{18}$$

subject to the nonnegativity assumption on Q, Δ^+, Δ^- and constraint (3).

The above model, hence, in the symmetric case is an optimization problem with goaling structure. For a review of goal programming, we refer the reader to Charnes and Cooper (1977). For applications of goal programming and nonlinear complementarity theory to energy markets, see Thore and Isser (1987).

2. QUALITATIVE RESULTS

In this Section we present the qualitative results for the competitive market model in the presence of transportation target goals developed in the preceding section. We first provide our existence result and then the uniqueness result. We establish that, under certain monotonicity conditions and nontrivial penalties, existence and uniqueness of the equilibrium pattern are guaranteed.

Theorem 2: Assume that the supply price and the demand price functions $\pi(s)$ and $-\rho(d)$ are monotone and that the transportation cost function $c(Q)$ is strongly monotone. Assume also that $M_{ij}^2+N_{ij}^2 \neq 0$ for all i, j. Then there exists a solution to (14).

Proof: Let $\bar{\Delta}_{ij}^+ = \bar{\Delta}_{ij}^- = 0$ and $\bar{Q}_{ij} = Q_{ij}^{\#}$. Then

$$\sum_{ij}(\pi_i(s) + c_{ij}(Q) - \rho_j(d)) \cdot (Q_{ij} - \bar{Q}_{ij}) + \sum_{ij}(M_{ij}(\Delta_{ij}^+ - \bar{\Delta}_{ij}^+) + N_{ij}(\Delta_{ij}^- - \bar{\Delta}_{ij}^-))$$

$$= \sum_{ij}(\pi_i(s) + c_{ij}(Q) - \rho_j(d)) \cdot (Q_{ij} - Q_{ij}^{\#}) + \sum_{ij} M_{ij}\Delta_{ij}^+ + N_{ij}\Delta_{ij}^-. \qquad (19)$$

Since $\|(\Delta^+, \Delta^-, Q)\|^2 = \sum_{ij} {\Delta_{ij}^+}^2 + \sum_{ij} {\Delta_{ij}^-}^2 + \sum_{ij} Q_{ij}^2$, if $\sum_{ij} Q_{ij}^2$ is bounded, while $\sum_{ij}({\Delta_{ij}^+}^2 + {\Delta_{ij}^-}^2)$ is sufficiently large, by the assumption that $M_{ij} \geq 0$,$N_{ij} \geq 0$ and $M_{ij}^2 + N_{ij}^2 \neq 0$, for all i, j, we have that

$$\sum_{ij}(\pi_i(s) + c_{ij}(Q) - \rho_j(d)) \cdot (Q_{ij} - Q_{ij}^{\#}) + \sum_{ij}(M_{ij}\Delta_{ij}^+ + N_{ij}\Delta_{ij}^-) \geq 0. \qquad (20)$$

On the other hand, if $\sum_{ij} Q_{ij}^2$ is not bounded, and is taken to be sufficiently large, then by the monotonicity assumption, and an application of Schwarz's inequality we conclude that

$$\sum_{ij}(\pi_i(s) + c_{ij}(Q) - \rho_j(d)) \cdot (Q_{ij} - Q_{ij}^{\#})$$

$$= \sum_{ij}((\pi_i(s) + c_{ij}(Q) - \rho_j(d)) - (\pi_i(s^{\#}) + c_{ij}(Q^{\#}) - \rho_j(d^{\#})) \cdot (Q_{ij} - Q_{ij}^{\#})$$

$$+ \sum_{ij}(\pi_i(s^{\#}) + c_{ij}(Q^{\#}) - \rho_j(d^{\#})) \cdot (Q_{ij} - Q_{ij}^{\#}) \geq \alpha \sum_{ij} (Q_{ij} - Q_{ij}^{\#})^2$$

$$-(\sum_{ij}(\pi_i(s^{\#}) + c_{ij}(Q^{\#}) - \rho_j(d^{\#}))^2)^{\frac{1}{2}} (\sum_{ij}(Q_{ij} - Q_{ij}^{\#})^2)^{\frac{1}{2}} > 0, \qquad (21)$$

where $s_i^{\#} = \sum_{j=1}^{n} Q_{ij}^{\#}$, $d_j^{\#} = \sum_{i=1}^{m} Q_{ij}^{\#}$, and α is a positive parameter used in the definition of strong monotonicity.

Therefore, the variational inequality satisfies the coercivity condition of a theorem of More' (1974), from which existence of a solution to the variational inequality then follows.

Theorem 3: Under the same conditions as imposed in Theorem 3, uniqueness of the solution pattern is also guaranteed.

Proof: Suppose that (Δ^+, Δ^-, Q) and $(\bar{\Delta}^+, \bar{\Delta}^-, \bar{Q})$ are two distinct solutions of the variational inequality (14). Then

$$\sum_{ij}(\pi_i(s)+c_{ij}(Q)-\rho_j(d))\cdot(\bar{Q}_{ij}-Q_{ij})+\sum_{ij} M_{ij}(\bar{\Delta}_{ij}^+-\Delta_{ij}^+)+\sum_{ij} N_{ij}(\bar{\Delta}_{ij}^- - \Delta_{ij}) \geq 0 \tag{22}$$

$$\sum_{ij}(\pi_i(\bar{s})+c_{ij}(\bar{Q})-\rho_j(\bar{d}))\cdot(Q_{ij}-\bar{Q}_{ij})+\sum_{ij} M_{ij}(\Delta_{ij}^+-\bar{\Delta}_{ij}^+)+\sum_{ij} N_{ij}(\Delta_{ij}^- - \bar{\Delta}_{ij}^-) \geq 0. \tag{23}$$

Adding then (22) and (23), we obtain

$$\sum_{ij}(\pi_i(\bar{s}) + c_{ij}(\bar{Q}) - \rho_j(\bar{d}) - (\pi_i(s) + c_{ij}(Q) - \rho_j(d))) \cdot (Q_{ij} - \bar{Q}_{ij}) \geq 0. \tag{24}$$

But, under the strong monotonicity assumption on $c(Q)$ and monotonicity of $\pi(s), -\rho(d)$, we conclude from (24) that $Q_{ij} = \bar{Q}_{ij}$, for all i and j.

Therefore, from (22) and (23) we have that

$$\sum_{ij} M_{ij}(\bar{\Delta}_{ij}^+ - \Delta_{ij}^+) + \sum_{ij} N_{ij}(\bar{\Delta}_{ij}^- - \Delta_{ij}^-) \geq 0 \tag{25}$$

$$\sum_{ij} M_{ij}(\Delta_{ij}^+ - \bar{\Delta}_{ij}^+) + \sum_{ij} N_{ij}(\Delta_{ij}^- - \bar{\Delta}_{ij}^-) \geq 0. \tag{26}$$

Also, we have that

$$\bar{\Delta}_{ij}^+ - \bar{\Delta}_{ij}^- = \bar{Q}_{ij} - Q_{ij}^{\#} = Q_{ij} - Q_{ij}^{\#} = \Delta_{ij}^+ - \Delta_{ij}^-. \tag{27}$$

From (25) and (22), specifying to (i,j) in variational inequality (14) by setting $Q'_{kl} = Q_{kl}$, $\Delta_{kl}^{+'}=\Delta_{kl}^+$, $\Delta_{kl}^{-'} = \Delta'_{kl}$, for all $kl \neq ij$ and setting $Q'_{ij} = \bar{Q}_{ij}$, $\Delta_{ij}^{+'} = \bar{\Delta}_{ij}^+$, $\Delta'_{ij} = \bar{\Delta}_{ij}$, we obtain: $(M_{ij}+N_{ij})\cdot(\bar{\Delta}_{ij}^+ - \Delta_{ij}^+) \geq 0$. Similarly, from (26) and (23), specifying to (i,j) in variational inequality (14) by setting $Q'_{kl} = \bar{Q}_{kl}$,$\Delta_{kl}^{+'} = \bar{\Delta}_{kl}^+$, $\Delta_{kl}^{-'} = \bar{\Delta}_{kl}^-$, for all $kl \neq ij$, and $Q'_{ij} = Q_{ij}$, $\Delta_{ij}^{+'} = \Delta_{ij}^+$, and $\Delta'_{ij} = \Delta_{ij}^-$, we obtain: $(M_{ij}+N_{ij})\cdot(\Delta^+_{ij}-\bar{\Delta}^+_{ij})\geq 0$. Hence, $\bar{\Delta}^+_{ij}=\Delta^+_{ij}$, $\forall\, i,j$ and it thus follows that $\bar{\Delta}^-_{ij}=\Delta^-_{ij}$, $\forall\, i,j$.

3. THE ALGORITHM

In this Section we propose an algorithm for the computation of the equilibrium production, consumption, trade, and overshipment/undershipment pattern for the market model developed in Section 1. The algorithm converges under the assumption that the supply price π and demand price ρ functions are monotone and that the transaction cost function c is strongly monotone. As established in Section 2, under such conditions the equilibrium pattern exists, and is unique.

The algorithm is a new version of the method of multipliers (cf. Pan (1992)). We view its advantage as lying in its splitting feature. In particular, it resolves the variational inequality problem (14) into a series of two simpler variational inequality subproblems, with only nonnegativity constraints. The first subproblem takes the form of the variational inequality governing the well-known spatial price equilibrium problem, for which numerous efficient algorithms exist (see, e.g., Nagurney (1987), Nagurney and Zhao (1991), and the references therein). The second variational inequality subproblem is a very simple linear problem in the (Δ^+,Δ^-) variables only. It, in turn, can be solved via a Gauss-Seidel decomposition method described in Nagurney (1987). We now state the algorithm as follows:

Given a sequence of parameters $0<r_0\leq r_1\leq r_2\leq\ldots$,

Step 0: Set the iteration count $\tau=0$. Initialize $({\Delta^+_{ij}}^0,{\Delta^-_{ij}}^0)$, and λ^0_{ij}, for $i=1,\ldots,m$, $j=1,\ldots,n$.

Step 1: Compute $(s^{\tau+1},Q^{\tau+1},d^{\tau+1})$, satisfying (1) and (2), such that:

$$\sum_{i=1}^{m}\pi_i(s^{\tau+1})\cdot(s'_i-s^{\tau+1}_i)-\sum_{j=1}^{n}\rho_j(d^{\tau+1})\cdot(d'_j-d^{\tau+1}_j)$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}(Q^{\tau+1})+r_\tau(Q^{\tau+1}-(\Delta^+_{ij})^\tau+(\Delta^-_{ij})^\tau-Q^\#_{ij})-\lambda^\tau_{ij})\cdot(Q'_{ij}-Q^{\tau+1}_{ij})\geq 0$$

$$\forall Q'_{ij}\geq 0,\ s'_i=\sum_{j=1}^{n}Q'_{ij},\ d'_j=\sum_{i=1}^{m}Q'_{ij},\ \forall i,j. \tag{28}$$

Step 2: Compute $((\Delta^+)^{\tau+1},(\Delta^-)^{\tau+1})\geq 0$ such that:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(M_{ij}-r_\tau(Q^{\tau+1}_{ij}-(\Delta^+_{ij})^{\tau+1}+(\Delta^-_{ij})^{\tau+1}-Q^\#_{ij})+\lambda^\tau_{ij})\cdot({\Delta^+_{ij}}'-(\Delta^+_{ij})^{\tau+1})$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}(N_{ij}+r_\tau(Q_{ij}^{\tau+1}-(\Delta_{ij}^{+})^{\tau+1}+(\Delta_{ij}^{-})^{\tau+1}-Q_{ij}^{\#})-\lambda_{ij}^{\tau})\cdot(\Delta_{ij}^{-\prime}-(\Delta_{ij})^{\tau+1})\geq 0 \tag{29}$$

$$\forall\Delta_{ij}^{+\prime}\geq 0,\ \Delta_{ij}^{-\prime}\geq 0,\ i=1,\ldots,m;\ j=1,\ldots,n.$$

Let $\lambda_{ij}^{\tau+1}=\lambda_{ij}^{\tau}+r_\tau(Q_{ij}^{\#}-Q_{ij}^{\tau+1}+(\Delta_{ij}^{+})^{\tau+1}-(\Delta_{ij}^{-})^{\tau+1})$.

Step 3:

If convergence has been reached within a prespecified tolerance ϵ, then stop; otherwise, set $\tau=\tau+1$, and go to Step 1.

Notice that in Steps 1 and 2, the respective subproblems are strongly monotone, and, hence, the sequence of iterates $s^{\tau+1}, Q^{\tau+1}, d^{\tau+1}, (\Delta^{+})^{\tau+1}, (\Delta^{-})^{\tau+1}$ is well-defined.

4. NUMERICAL RESULTS

In this Section we describe the results of our numerical experimentation for the algorithm presented in Section 3. The problems we considered ranged in size from ten supply markets and ten demand markets through fifty supply markets and fifty demand markets. The results of the numerical experimentation are reported in Table 1.

The market equilibrium problems were generated in the following manner. We considered three sets of problems - in the first set, the transportation goal targets $Q_{ij}^{\#}$, $i=1,\ldots,m$; $j=1,\ldots,n$ were generated randomly and uniformly in the range $[10,50]$; in the second set, the transportation targets were set equal to 10 for all transportation links, and in the third set, the transportation targets were set equal to 50 for all links. The corresponding results are reported, respectively, in sets one to three of Table 1. The penalty terms M_{ij} and N_{ij} were also generated randomly and uniformly in the ranges $[1,10]$. In the first set of examples, however, we set all of the $N_{ij}=0$ to model negative economic externalities resulting from overshipment along the transportation links.

The supply price functions, $\pi(s)$, demand price functions $\rho(d)$ and transportation cost functions $c(Q)$ were all linear, repspectively, of the form:

$$\pi_i(s)=\sum_{l=1}^{m}r_{il}s_l+t_l,\quad \rho_j(d)=-\sum_{k=1}^{n}m_{jk}d_k+q_j,\quad c_{ij}(Q)=\sum_{k=1}^{m}\sum_{l=1}^{n}g_{ijkl}Q_{kl}+h_{ij}. \tag{30}$$

The Jacobians of the supply price, demand price, and transportation cost functions were asymmetric.

The above functions were generated randomly and uniformly in the ranges as follows: $r_{ii}\in[3,10]$, $t_i\in[10,25]$; $m_{jj}\in[1,5]$, $q_j\in[150,650]$; $g_{ijij}\in[1,15]$, $h_{ij}\in[10,25]$; with the cross-terms r_{il}, m_{jk}, and g_{ijkl}, generated so that the

Jacobians were strictly diagonal dominant, thereby, guaranteeing existence and uniqueness of the equilibrium pattern.

The same function ranges had been used in the market examples solved in Nagurney and Zhao (1991). The number of cross-terms in each of the examples was, furthermore, set equal to five.

The termination criterion used was that the equilibrium conditions for each variational inequality subproblem (cf. (28) and (29)) held within a tolerance of .5, and that $|Q_{ij}^{t+1} - Q_{ij}^{t}| \leq .005$, for all i, j, $|(\Delta_i^+)^{t+1} - (\Delta_i^+)^t| \leq .005$, for all i, and $|(\Delta_i^-)^{t+1} - (\Delta_i^-)^t| \leq .005$, for all i. The parameters r_t were set as follows: $r_0 = 1$, $r_t = r_0 \times 2^t$.

We used a Gauss-Seidel algorithm described in Nagurney (1987) for the computation of the solution to subproblem (28) in which we, however, embedded the exact demand market equilibration algorithm developed in Dafermos and Nagurney (1989), to fully exploit the underlying network structure of the spatial price equilibrium problem. To solve the subproblem (29) we also applied a Gauss-Seidel algorithm, solving the subproblem term by term.

The algorithm was coded in FORTRAN and the system used for all of the numerical work was the IBM 3090-600J at the Cornell National Supercomputer Facility.

In Table 1 we report for each set of examples the following data: the CPU time (without input/output times), the number of iterations t, the number of transportation links with $\Delta_{ij}^- > 0$, the number of transportation links with $\Delta_{ij}^+ > 0$, the minimum absolute deviation $|Q_{ij} - Q_{ij}^{\#}|$, the maximum absolute deviation $|Q_{ij} - Q_{ij}^{\#}|$, and the average absolute deviation $\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}|Q_{ij} - Q_{ij}^{\#}|}{mn}$.

As can be seen from Table 1, the algorithm converged for all of the examples, requiring, at most, only several seconds of CPU time. In the first set of examples, with $Q_{ij}^{\#} \in [10, 50]$, in any given example and with all $N_{ij} = 0$, the number of transportation links with $\Delta_{ij}^+ > 0$ was negligible. This is to be contrasted with the second set, in which $Q_{ij}^{\#} = 10$, and the N_{ij}'s were generated to lie in the range $[1, 10]$; that is, the fixed target was set rather low, in which case there were many more markets with $\Delta_{ij}^+ > 0$. On the other hand, in the third set of examples, since the $Q_{ij}^{\#}$'s were set at 50, the majority of the transportation links had undershipments as reflected in the fact that many more links had the respective $\Delta_{ij}^- > 0$.

5. SUMMARY AND CONCLUSIONS

In this paper we have described how some of the ideas set forth in classical goal programming can be synthesized with variational inequality problems. In particular, we introduced a new competitive spatial market equilibrium model which permits regulators to assign transportation goal targets. Overshipment /

undershipment is then penalized according to preassigned penalties. The model captures mathematically certain negative economic externalities and opens the door for the development of other models, as well.

We first established that the governing equilibrium conditions can be formulated as a variational inequality problem. We then proved that under certain monotonicity conditions the equilibrium problem is guaranteed to have a solution and the solution is unique.

We then proposed an algorithm, developed for a class of variational inequalities with linear constraints, of which our model is a member of. The algorithm decomposes the variational inequality into two simpler variational inequality subproblems, the first of which has the notable feature that it is identical to the variational inequality governing the well-known spatial price equilibrium problem. The algorithm under the same conditions that guarantee existence and uniqueness of the equilibrium pattern is guaranteed to converge. Numerical results were then presented for fifteen examples, which highlighted different modelling features.

This research may be viewed as a contribution to the growing literature on mathematical programming approaches to policy interventions in spatial market models.

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Table 1: Numerical Results for the Variational Inequality Algorithm for Spatial Markets with Transportation Targets

m	n	$M \in [1,10]$, $N = 0$, $Q^{\#} \in [10,50]$					
10	10	.048(8)	100;	0;	1.5;	45.81;	26.73
20	20	.302(8)	400;	0;	1.29;	48.38;	31.67
30	30	.824(10	897;	3;	7.13;	48.92;	27.50
40	40	2.60(10)	1599;	1;	1.09;	48.29;	28.23
50	50	6.15(10)	2499;	1;	3.91;	47.44;	30.08
m	n	$M \in [1,10]$, $N \in [1,10]$, $Q^{\#} = 10$					
10	10	.05(9)	87;	9;	0.;	16.19;	7.79
20	20	.38(10)	383;	6;	0.;	10.00;	8.61
30	30	.93(10)	884;	10;	0.;	18.71;	9.04
40	40	3.28(10)	1592;	3;	0.;	17.83;	9.39
50	50	7.10(10)	2495;	2;	0.;	10.00;	9.58
m	n	$M \in [1,10]$, $N \in [1,10]$, $Q^{\#} = 50$					
10	10	.048(8)	100;	0;	21.70;	50.00;	47.08
20	20	.344(8)	400;	0;	26.47;	50.00;	48.43
30	30	.854(10)	900;	0;	18.18;	50.00;	48.95
40	40	2.89(10)	1000;	0;	20.25;	50.00	49.36
50	50	6.56(9)	2500;	0;	33.05;	50.00;	49.53

* CPU time in sec. (τ), # of Δ_{ij}^{-}'s > 0; # of Δ_{ij}^{+}'s > 0; min. abs. dev.; max. abs. dev.; ave. abs. dev.