CITIES' POWER LAWS OF WELFARE AND LAND RENTS: A MICROECONOMIC THEORY

ABSTRACT

The power law is a fundamental constant in the dynamics of organisms and ecological systems. That cities exhibit this law for its dynamics is the surprising evidence from empirical studies developed recently. Applying extreme value theory to describe all urban agents' rational stochastic behavior and Alonso's urban economic principles, this paper formulates a theoretical construct of urban systems that supports the evidence of power laws. The observed super-linear increase of rents and wealth emerges here from microeconomic behavior due to population's diversity and size. A dynamic model of cities' growth emerges allowing to predict the expected evolution of urban organizations.

Key words: power law, returns to scale, random bids auctions, urban dynamics.
JEL codes: D03, D60, D90, R11
Total number of words: 6542

1. INTRODUCTION

The existence of a general law that governs the dynamics of all cities has been present in geography since the early evidence that cities complies with the Zipf's law (see Gabaix, 1999). More recently, however another regularity has been postulated: the power law. In several papers West, Bettencourt and colleagues (W-B papers: Bettencourt et al., 2007; 2008) report studies of a number of cities across the developed world that support the argument that cities, like other organisms in nature, consume and produce according to a power law with population *N*: $y = N^{\beta}$. This result is striking because for the first time the evolution of cities resembles a basic evolutionary law of organisms, i.e. social structures replicating known natural processes, or rational beings replicating the basic structure of "less intelligent" creatures. Thus, there appears to be important lessons to learn from what is known about the evolution of species to help understanding the dynamic of cities. For example, that under some circumstances power laws in organisms lead to catastrophic ends at critic levels of development. But it also raises the fundamental question about the underlying mechanism that yields such invariability in social organizations with such different histories and cultures.

An interesting and relevant lesson from biology comes from the once enigmatic parameter β in organisms (see West 1999), empirically observed as a multiple of ¼ in almost all forms of life, defining metabolic rate (energy consumption), time scale (e.g. lifespan and heart rate) and sizes (aorta lengths or tree heights). A theoretical explanation emerged using fractal geometry and the fact that the fractal structure at a micro scale ends in a common dimension given by the size of the organism's cell (West et al. 1999; 2001).

The evidence provided by W-B was obtained from a data of numerous urban indicators of American, European and Chinese cities revealing the statistically significant relationship between several indicators, denoted y, and population (N) along time t: $y(t) = y_0 N(t)^{\beta}$; here, y_0 represents a normalization constant that varies across cities. Their main conclusions are: first, that for every urban indicator the scale parameter β is statistically constant for the set of cities analyzed; second, that indicators of economic quantities that characterize the creation of wealth and innovation show increasing returns of scale, i.e. $\beta > 1$; third, that indicators of material infrastructure are characterized by economies of scale, i.e. $\beta < 1$. Combined, these results justify a strong tendency to concentrate population on large cities: per capita wealth increases while living costs (of infrastructure) decrease.

The W-B evidence directly motivates the question if the observed urban power law can be derived similarly to organisms from allometric -size and shape- relations. While this is a line of research, is not the approach followed of this paper. Instead, in what follows I use microeconomic principles to formulate a theory of the microscopic social and economic behavior of all agents in the city, i.e. at the individual level, including their interactions in all goods markets and their social life. From this theory emerges an explanation for power laws in cities between land prices and welfare and the population size that encapsulates and combines two potential sources of scale effects: economies of scale in the urban economy, which is the classical effect studied by economists, and the effect of the greater spectrum of opportunities offered for creativity and innovative production by larger and more diverse population.

The herein proposed theory of the cities' dynamics stands on one fundamental assumption: humans are rational beings facing stochastic information of the environment, which is modeled as if all agents in the city behave *á la* McFadden (Domencic and McFadden, 1975) maximizing a random utility or profit (Section 2), and landowners maximize rents *á la* Alonso (1964) in a context where bids for land are also stochastic (Section 3). It is precisely from the randomness of this optimizing behavior that the power law of rents and wealth with population emerges, with an explanation for the scale parameter beta directly from the variance of the extreme value distribution (Type II or Fréchet) that characterizes the stochastic distribution of bids in the auction of urban land. This model naturally yields a model of urban growth of Section 4 and therefore relevant consequences on the city dynamics. Additionally, under the weak assumption of non-decreasing scale economies in production and consumption, the scale parameter derived from agents heterogeneity supports –on its own- super-linear returns on the population scale, while scale economies –if any- reinforces this effect.

2. TOWARDS A THEORY OF URBAN SOCIAL ORGANIZATIONS

The complex behavior of agents in the city, including individuals' socio-economic activities and firms' business activities, is the matter of this Section assuming that all agents have a known residence location. This approach benefits from the significant literature on urban transport modeling, summarized for example by Ortúzar and Willumsen (2001) and Ben-Akiva and Lerman (1985). A brief summary of the essentials of this literature plus an additional hierarchical structure on the decision process, a it what follows.

The human organization is described here by the individual behavior of their members, including households and firms, who behave similarly maximizing a measure of satisfaction (utility or profit) under constrained resources. In this context we consider a large number of heterogeneous agents that interact creating socio-economic structures that emerge from their mutual social and economic interdependency and their maximizing behavior in a spatial context.

2.1 The Gumbel utility model

Proposition 1 (Gumbel stochastic utilities): Agent's utilities are independent random variables $\omega \sim Gumbel(\mu, v)$, with $\mu, v \in \mathbb{R}$.

The merits of the independent Gumbel distribution of benefits comes from the Extremal Types Theorem (Fisher and Trippett 1928, generalized by Gnedenko, 1943). They proved that the asymptotic (no-degenerative) distribution functions of the maximum of a set of independent stochastic variates belongs to a set of three types of extreme values distributions (d.f.), no matter what are the d.f. of the maximized variates. See also Leadbetter, et al. (1983); Galambos (1987), Mattson et al. (2011) for references. Hence, an extreme value distribution is –by the extremal types theorem- to the *maximum* operator, as the normal distribution is -by the central limit theorem- to the *addition* operator.

The Type I, *Gumbel* or double exponential d.f. is particularly relevant in utility maximizing models because out of the three extreme value distributions, this one is the attractor of a larger number of maximized d.f. and its support is the real numbers, which is consistent with the definition of utilities in economics as an ordinal. The Gumbel d.f. became popular with the individual's discrete choice theory with random utilities proposed by Domencic and McFadden (1975), whom under the assumption of identical and independent Gumbel variates, proved that the probability that a given alternative $i \epsilon C$ yields the maximum utility is given by the now popular multinomial logit model: $P_i = \frac{e^{\mu v_i}}{\sum_{j \epsilon C} e^{\mu v_j}}$. Another well-known *Gumbel* based model is the nested logit model (with a similar close probability function), which is of interest in this paper because of the hierarchical choice process considered next.

2.2 The individuals and households choice process

Each individual agent is assumed to choose, conditional on the residential location, the optimal set of discrete activities and their locations, along with the required set of consumption of goods and expenditure of time; on these choices she allocates the limited income and time as to maximize utility. This complex space of individual's options and the choice process is simplified as a hierarchical process as shown in Figure 1.

Consider an urban area partitioned in a set *I* of locations indexed by i, and consider one inhabitant indexed by $n \varepsilon C_H$ that belongs to a household indexed by h. At any given time the individual faces a set *K* of leisure (social) and productive (work) activities available in the region, indexed by $k \in K$. Following the hierarchical choice process of Figure 1, provided that individual *n* resides at location *i,* she chooses activity k -described by the random utility ω_{nik}- with probability P_{nik}; conditional on this choice the individual compares the location to perform this activity and chooses location j with probability P_{nikj} as the optimal location which yields her a utility ω_{nikj}.

Under the assumption that rational agents choose among options that maximize her utility, we define $\omega_{nik} = max_{j \in I}(\omega_{nikj})$ and $\omega_{ni} = max_{k \in k}(\omega_{nik})$. Assuming a sufficiently large set of locations *I,* no matter what are the distributions of the lowest level utilities (ω_{nikj}), as long as they are independent we know that by the Extremal Types Theorem each ω_{nik} distribute extremal. Additionally, since utilities are unbounded real numbers such variates distribute asymptotically Gumbel; because the Gumbel is close to the maximization operator then ω_{ni} is also Gumbel. This simple hierarchical structure of random utilities may also include additional random terms (with some specific form) yielding a nested logit probability for each choice in the hierarchy (see Ben-Akiva y Lerman, 1985).

In this choice framework a static equilibrium is attained in all markets in the urban system yielding goods prices, labor wages and transport times and costs that are inputs for the consumers' estimation of utilities; this internalizes into individuals' utilities market signals. Equilibrium prices are differentiated by spatial location according to transport costs that spatially differentiate otherwise non-differentiated goods. Consumers' utilities are also affected by travel times further differentiating consumption across the urban space. Additionally, leisure activities involve not only consumption but also social interactions, therefore the utility derived also depends on travel costs which are conditional on the agent's location. In sum, utilities considered to make optimal choices are evaluated at the urban market equilibrium which yields optimal activities and expenditure –of time and income- conditional on the agent's location.

Regarding the emergence of power laws, what is relevant of this choice process is the assumption of the irrelevance of the d.f. at the micro choice level utilities ω_{nikj}, the only relevant assumption is that they are independent variables, enough to conclude that the individual's maximum utility ω_{ni} is distributed independent Gumbel. Also relevant is that this utility is conditional on the residential location and that it can be computed for any location in the urban area. Additionally, this distribution is preserved from individuals to households units under the additional assumption that households distribute their common resources in other to maximize the global utility of the household unit, given the residential location and the maximum utility attainable by each member of the household from their optimal choice process.

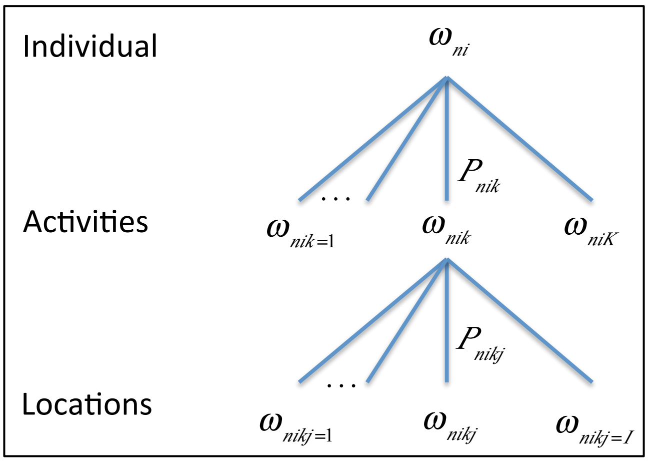

Figure 1: Hierarchical individuals' choice process

2.3 The firms' behavior

Firms' behavior in the urban system can be modeled analogously to individuals and households. In the same setting, consider firms as indexed by $n \in C_F$ located at i. The firms' objective is to maximize a utility or profit by choosing what and how much to produce among a finite set of production options, each one defined by inputs and outputs of goods' interactions with other firms and by the consumers' labor and consumption, all spatially distributed. Following Figure 1, the bottom level represents the discrete set of spatial interaction options, each one yielding different levels of profit.

As before, what matters is that at the top level of the decision tree, there is a random utility or profit conditional on the location choice that is independent Gumbel distributed. For this to hold, considering the Extremal Type theorem, it is enough to assume that at the micro detailed choice level the utilities ω_{nikj} are stochastic independent variables.

In this hierarchical choice process again market prices of inputs and outputs feed profit levels, as well transport costs and freight times. Industries may face economies of scale and scope and the market may be totally or partially competitive, all options that define optimal production and market equilibrium without affecting the conclusion of Gumbel profits.

2.4 Section Remarks

The above model of decision making leads to the conclusion that households' and firms' choice making processes can be represented by a set of generic agents that make optimal choices in large sets of options of social and economic activities spatially distrubuted. It is worth noting that all these choices are interdependent through market interactions between consumers, between suppliers, and between consumers and suppliers. Out of these interactions price signals emerge in the economy as the result of a complex non-linear system equilibrium that is implicit in this approach. An example of a detailed model of these interactions in a similar random utility context can be seen in Anas and Liu (2007), where consumers and firms are modeled by Cobb-Douglas random utilities and profit functions.

As a result agents' utility conditional on the location choice define the vector households' utilities $(\omega_{hi})_{hi}$ and firms' profits $\left(\omega_{fi}\right)_{fi}$ as:

$$\omega = (\omega_{ni})_{ni} = ((\omega_{hi})_{hi}, \left(\omega_{fi}\right)_{fi}; \; h = 1, \dots, N_h, f = 1, \dots, N_f, i\epsilon I).$$

where each ω_{ni} is an independent *Gumbel* variates (μ_n, v_{ni}). The mode of the utility distribution, v_{ni}, represents the expectation of the utility the agent can obtain from the optimal choice process in the urban economy conditional on her location; parameter μ is the variance of these utilities.

Studies of expected utility out of the agents' behavior assuming the *Gumbel* distribution are common in the transport literature that followed Daniel McFadden's legacy, and models already applied in a large number of cities can provide estimates of the conditional utility vector. As we shall see below, it is of special interest to study the existence of scale economies by testing if at the agent level, benefits and profits scale with population of the city *N,* i.e. if $\omega_{ni}(N)$.

This section has only sketched a model of agents' activities in a social and economic system, with the aim of providing a complete behavioral model structure of social organizations. Although it is a simple construct, it does contribute to explain the emergence of power laws in the next section by linking the results obtained there with the individuals' micro choice processes.

3. THE LOCATION PROCESS

In this section the Alonso's urban economic principles are considered to allocate the urban land to different residential and non-residential agents following the best bidder rule. Each of these agents' behavior is represented by their bids for location options, derived directly from the indirect utility conditional on the location choice.

An illustrative example of how random bids are derived from random utilities follows. Consider Anas and Liu (2007)'s general equilibrium model where the Cobb-Douglas utility of residents yields the following logarithmic indirect utility function conditional on residential location: $U_{ni}(R) = a_{ni} - b_n \cdot ln(R_i) + \varepsilon_{ni}$, with an additive stochastic term distributed independent *Gumbel*; *U* is utility, *R* is land price per square area unit, *a* and *b* are parameters. The maximum willingness to pay or bid (θ_{ni}) to attain a given level of utility is derived by Rosen (1974) inverting *U(R).* Note that in our case with stochastic utilities, setting a given utility level at U_n implies a constant value of utility for every realization of the random term, which given that *a* and *b* are constants, we require that bids offset the variability of ε_{ni}. Thus, inverting the utility function in rents yields the following stochastic bid: $\theta_{ni} = \exp\left(\frac{a_{ni}-U_n}{b_n}\right) \cdot \xi_{ni}$, with $\xi_{ni} = exp\left(\frac{\varepsilon_{ni}}{b_n}\right)$; hence, $\theta_{ni} \epsilon \mathbb{R}_{++}$. Notice the one to one mapping between random terms (ε_{ni}) and (ξ_{ni}) exactly offsets the effect of the variability of utilities on bids to hold the utility constant across all optional locations.

Given that ε_{ni} are independent *Gumbel* variates, as assumed in the previous section, it directly follows that the stochastic terms ξ_{ni} are distributed independent *Fréchet,* as shown in Mattssonn et al. (2011). Therefore, we conclude this preamble noting that models of utility with logarithmic rents yield positive bids that, combined with *Gumbel* distributed utilities, support the assumption of *Fréchet* distributed bids.

3.1 The *Fréchet* model for land rents

Consider again the urban area partitioned in a set *I* of locations indexed *i*, with population of N_h households and an economy with N_f firms, all agents indexed by *n*, with $N_a=N_h+N_f$.

Assumption 2 (Fréchet stochastic bids): *Agent's bids for urban land are independent random variables* $\theta_{ni} \sim Fréchet(\beta, v_{ni})$, *with* $\theta_{ni} \epsilon \mathbb{R}_{++}, v_{ni} > 0, n \epsilon C, i \epsilon I,$ *and* $\beta > 0$.

β is the scale parameter inversely related with the variance of the distribution. The *Fréchet*, also called Type II extreme value distribution function, is the domain of attraction or the limit non-degenerative distribution of the maximum operator of a set of variables distributed independent *Fréchet*, among other distributions; i.e. this d.f. is closed to the maximum operator.

Then, Assumption 2 holds if bids are the result of the maximization of underlying variables with independent *Fréchet* distributions. In the case of bids, this holds because they represent the maximum amount the agent is willingness to pay to attain a given level of utility. Then, for Assumption 2 to hold it is sufficient to assume that willingness to pay are positive independent *Fréchet* stochastic variables because their maximum is asymptotically positive and distributed independent *Fréchet*.

It is worth noting here that the ξ-*Fréchet* variates of the location bids are functionally dependent on ε-*Gumbel* variates of utilities (see Section 2). This implies that their variances are also dependent, such that $\beta = f(\mu)$ with $\mu = (\mu_n, \forall n \epsilon C)$. The importance of this fact will be evident below when we interpret the power law emerging from micro behavior. It is also important that the assumption of independent *Fréchet* variates refers to independent bids between agents.

Assumption 3 (Auction): *The land market is auctioned to the best bidder.*

Following Von Thünen (1863), and particularly after Alonso's (1964) seminal work, urban economists have hold on and develop the basics of urban economics from this assumption. The auction as a trade protocol in land markets is a direct consequence of differentiated goods (Rosen, 1974), because the specific location of land gives the owner the right to enjoy the neighboring amenities which are different from one location to another. Land, however, is a special case of differentiated goods, because every land lot has close substitutes as it shares neighbor attributes with close locations, hence information about expected prices is considered commonly available to every agent. This, however, does not make them homogeneous goods, but classifies urban land markets for the type of common value auctions (McAfee and McMillan, 1987).

Ellickson (1981) uses the auction protocol to support that land bids are extreme value variables, because he asserts that only the maximum bid of a socioeconomic cluster is relevant for the auction. Hence no matter the d.f. of elementary bids within the cluster, the maximum bid relevant to the auction is an extreme variable. However, he ignored the positive domain of bids assuming a Gumbel distribution, widely used thereafter leading to multinomial logit models. Nevertheless, his argument is also valid to support the *Fréchet* model used hereafter.

Proposition 1 (Rents power law): *Under assumptions 2 and 3, the total rent of the city scales with population as a power law describing increasing returns to scale.*

Proof: Consider Assumption 2 to define the maximum bid as $\theta_i = max_n \theta_{ni}$ with $\theta_i \sim Fréchet(\beta, r_i)$, with $\beta > 1$[1] and $r_i > 0$ calculated, following Mattsson et al. (2011), as:

$$r_i = \left(\sum_{n \epsilon C} \mathrm{v}_{ni}{}^{\beta}\right)^{1/\beta} \qquad \forall i \in I \tag{1}$$

The expected rent yield by the auction (Assumption 3), resulting by the expected maximum bid, is computed by $R_i = K \cdot r_i$, with $K = \Gamma\left(1 - \frac{1}{\beta}\right)$ a known constant . It can be approximated by considering $v_{ni} \approx v_i$ the average bid across agents:

$$R_i \approx K \cdot \left(\sum_{n \epsilon C} \mathrm{v}_i{}^{\beta}\right)^{\frac{1}{\beta}} = \mathrm{v}_i \cdot K \cdot \left(\sum_{n \epsilon C} 1\right)^{\frac{1}{\beta}} = \mathrm{v}_i \cdot K \cdot N_a{}^{\frac{1}{\beta}} \tag{2}$$

which indicates that the expected rent at every location scales with N_a following a power law, i.e. *ceteris paribus* rents increase with population. It is direct to conclude that this law is sub-linear because $\frac{1}{\beta} < 1$. Note also that rents decrease with β, or increase with bids' variance.

---

[1] Note that the *Fréchet* distribution is defined for $\beta > 0$ but the mean is only defined for $\beta > 1$.

Using (2), replace the number of agents by total population *N* such that $N_a = aN, a \leq 1$, to compute the aggregate land rent as:

$$R = \sum_{i\epsilon I} R_i \approx K \cdot N_a^{\frac{1}{\beta}} \cdot \sum_{i\epsilon I} v_i$$

then

$$R \approx a^{\frac{1+\beta}{\beta}} K \cdot V \cdot N^{\frac{1+\beta}{\beta}} \qquad (3)$$

with $V = \frac{1}{N_a} \sum_{i\epsilon I} v_i$, the average bid value across locations.

Since $\beta > 1$, then $\frac{1+\beta}{\beta} \epsilon (1,2)$ implying that total rents have increasing returns to scale with population, i.e. total rents increase super-linearly with population. ■

Remark 1: Empirical support. From Assumption 2, the rent at any location is described by the stochastic variable $\theta_i = r_i \cdot \xi_i$, with ξ_i also distributed independent $Fréchet(\beta, 1)$. It follows that total land rents are also stochastic variables distributed *Fréchet* $(\beta, R/K)$. The multiplicative form of the stochastic term ξ_i implies that rents are heteroscedastic variables, which is strongly supported by evidence in several empirical of hedonic price models (see Cho, et al. 2008).

Remark 2: Super linearity emerges from microscopic diversity. The power law of equation (3) describes returns to scale produced by the variance of agents' behavior across the population –represented by $1/\beta$-, which in turn depends on the variance of agents' utilities and profits (proportional to $1/\mu$); i.e. β emerges from microscopic behavior. This effect is interpreted as caused by the better likelihood of larger and more diversified populations of making matching between the individual agents' optimal interactions.

Remark 3: Economies of scale. The stochastic super-linear effect is in addition to other potential "deterministic" economies of scale embedded in the expected average land value, i.e. *V=V(N)*. In the urban context these economies combine the consumers' socio-economic returns to scale (utilities), and business returns to scale in production (profit). Most studies in urban economics are focused in this relationship only. If there is any, these returns are built by individual agents' (households and firms) benefits scaling with population; i.e. $v_{ni} = v_{ni}(N)$. On this matter, urban economist consider the simple monocentric city –fix location of firms in the city center and workers in the periphery- to explains residential rents by transport costs, concluding that they scale up to $V(N) \propto N^{3/2}$ (Black and Henderson, 1999). But introducing relocation of residents and jobs and allowing different transport modes and route choices, reduces increasing travel with size down closer to a linear relationship with population (Anas and Hiramatsu, 2012). On the business sector, if production has constant returns to scale and face no external economies, the classical results is zero profit in the industry, i.e. land bids are constant with production scale (Anas and Liu, 2007). In empirical studies Henderson (1986) found evidence of intra-industry (location) economies of scale in some industries, particularly in medium size specialized cities, but they peter out as the city size increases; no relevant evidence of population (urban) scale economies was found. Then, the issue of scale economies is still under debate with evidence that tend to challenge the existence of significant economies of scale. Therefore, assuming $V(N) = b \cdot N^{\alpha}$ with b constant and $\alpha > 0$ is plausible. Under this conjecture equations (3) becomes:

$$R \approx A \cdot N^{\gamma} \qquad (4)$$

where A=$\alpha^{\frac{1+\beta}{\beta}} \cdot b \cdot K$ and $\gamma = \alpha + \frac{1+\beta}{\beta} > 1$. Note that even for an economy with dis-economies of scale up to of grade $\alpha > -\frac{1}{\beta}$, still $\gamma > 1$ and rents scale super linearly with population.

3.2 The *Fréchet* model of consumers' surplus

Assumption 4 (Equilibrium): *Every agent $n \epsilon C$ in the urban system is allocated somewhere in the set I, given that the number of agents in set C, denoted N_a, is equal or smaller than the size of set I.*

This definition of equilibrium, which clears demand but not necessarily supply of real estate units, is standard in urban economic literature. The special case is when supply also clears, which requires the additional condition that size of real estate stock equals total demand.

For Assumption 4 to hold each agent has to adjust their bids, either: upwards, if there is no location where the agent is best bidder until wins the auction in one location, thus reducing the utility level on which bids are conditional upon until reaches the maximum attainable utility; or downwards, if there several locations where the agents wins the auctions until there is only one, in this case increasing utility up to the attainable maximum. The analytical process of bids adjustments is discussed below (see *Remark 4*), meanwhile consider bids calculated at equilibrium utility levels.

Proposition 2 (Consumers' surplus): *Under assumptions 2, 3 and 4, and the following ratio definition of the n^{th} agent's surplus: $\phi_{ni} = \frac{\theta_{ni}}{R_i}$, at equilibrium the expected consumers' surplus of every agent is non-negative.*

Proof: The ratio defining ϕ_{ni} represents the proportion of the consumer willingness to pay capitalized into rents. Since R_i is a deterministic variable, from Assumption 2 the welfare defined by $\phi_{ni} = \frac{\theta_{ni}}{R_i}$ follows the distribution bids, i.e. independent $Fréchet(\beta, \frac{v_{ni}}{R_i})$, $n \epsilon C, i \epsilon I, v_{ni} > 0$. Then, the expected maximum surplus associated with the set *I* of available locations in the city is:

$$w_n = K \cdot \left(\sum_{i \epsilon I} \left(\frac{v_{ni}}{R_i} \right)^{\beta} \right)^{1/\beta} \qquad \forall n \in C \qquad (5)$$

Additionally, given a location *i* offered in the market, from Assumptions 2 and 3, agent *n* has the following probability of being the best bidder (Mattsson et al. 2011):

$$p_{ni} = \frac{\mathrm{v}_{ni}{}^{\beta}}{\sum_{m\epsilon C} \mathrm{v}_{mi}{}^{\beta}} = K^{\beta} \left(\frac{\mathrm{v}_{ni}}{R_i}\right)^{\beta} \quad \forall n \in C, i \in I. \tag{6}$$

Now, consider Assumption 4 to write the following agents' equilibrium conditions:

$$\sum_{i\epsilon I} p_{ni} = K^{\beta} \cdot \sum_{i\epsilon I} \left(\frac{\mathrm{v}_{ni}}{R_i}\right)^{\beta} \geq 1 \qquad \forall n \in C \tag{7}$$

Using (5) and (7) yields the conclusion that $w_n \geq 1, \forall n \in C$.

This results implies that the expected proportion of willingness to pay actually paid at equilibrium is either: $w_n = 1$, i.e. consumer's brake even because rents equal willingness to pay at the equilibrium location and all benefits are capitalized into rents; or $w_n > 1$, i.e. consumer retains a proportion of benefits, which holds, for example, when supply of real estate units exceeds demand. ■

Remark 4.Equilibrium bids and rents. Consider the case of total supply equal total demand where equation (7) holds for equality. Consider also a set of utility levels $(u_n)_n$ such that $\mathrm{v}_{ni} = u_n \cdot \bar{\mathrm{v}}_{ni}$ which adjust at equilibrium to comply with:

$$\sum_{i\epsilon I} p_{ni} = \sum_{i\epsilon I} \left(\frac{u_n \cdot \bar{\mathrm{v}}_{ni}}{\sum_{m \in C} u_m \cdot \bar{\mathrm{v}}_{mi}}\right)^{\beta} = 1 \qquad \forall n \in C \tag{8}$$

It is evident that this condition provides the following fixed point equation for equilibrium utilities:

$$u_n^* = \sum_{i\epsilon I} \left(\frac{\bar{\mathrm{v}}_{ni}}{\sum_{m \in C} u_m^* \cdot \bar{\mathrm{v}}_{mi}}\right)^{\beta} \qquad \forall n \in C \tag{9}$$

which once replaced in bids functions yields bids at equilibrium, and also rents:

$$R_i^* = K \cdot \left(\sum_{n\epsilon C} (u_n^* \cdot \bar{\mathrm{v}}_{ni})^{\beta}\right)^{1/\beta} \qquad \forall i \in I \tag{10}$$

A positive maximum consumers' surplus holds if total demand (size of set *C*) is equal or larger than the size of supply (size of set *I*). However, surplus becomes negative if total demand exceeds total supply, which generates homeless agents (see Martínez and Hurtubia, 2006). In the long run, the usual assumption is that total supply equals demand and all consumers brake even.

Theorem (Welfare power law): *At the city's long term market equilibrium, total welfare is given by the following power law:*

$$W \geq A \cdot N^{\gamma} \quad . \tag{11}$$

and for economies with non-negative economies of scale ($\gamma > 1$*) it represents a super-linear power law.*

Proof: Define the city's welfare as the sum of all agents' and real estate suppliers' surplus, with the latter given by rents. Proposition (2) proves that rents follow a super-linear power law with population size up to a degree of dis-economies of scale of $\alpha > -\frac{1}{\beta}$ (because $\beta > 1$). Then, $\alpha = 0$, complies with this condition.
Additionally, Proposition 2 proves that equation (11) represents a lower bound with two cases:

i) $\forall n \; w_n \geq 1$, in this case Propositions 1 and 2 demonstrate the theorem.
ii) $\exists n / w_n < 1$, i.e. some agents are homeless because are unable to outbid any other in the city; this case is irrelevant since the proof is conditional on the population in the city. ■

3.3. The base line

A final extension is necessary to complement the *Fréchet* model. Note that the rent equations (1) and (2) are homogeneous of degree one and the probability distribution of agents in space given by equation (6) is homogeneous of degree zero. Hence, the location model is invariant to a multiplicative factor on all bids (denoted by R_0), but all rents are amplified by R_0. Additionally, the general equilibrium framework used in this paper define relative prices and rents scaled by an unknown *numeraire price* represented in (12) by R_0. This *numeraire* factor modifies scale laws as follows:

$$\frac{R}{R_o} \approx A \cdot N^{\gamma} \tag{12}$$

$$\frac{W}{R_0} \geq A \cdot N^{\gamma} \quad . \tag{13}$$

This apparently simple correction to the scale law is relevant to properly compare dynamics among cities. The *numeraire* factor relates the prices in each urban system with the rest of the economy where it is embedded, it is common to all cities in the same economy but it might be different across economies or countries. Therefore this correction completes the power laws making them finally consistent with W-B evidence, comparable across different economies and consistent with microeconomic theory of equilibrium.

4. URBAN GROWTH

The power law of the city socioeconomics derived above embeds the dynamic process of the cities' growth described by Bettencourt, et al. (2007, 2008), with fundamental consequences on the evolution of urban systems. In this section their model is applied using the theory developed above.

The authors applies the power law associated to resources generation to the following simple balance equation on resources:

$$R(t) = E_o\, N(t)^{\rho} - E\, \frac{\partial N}{\partial t} \tag{14}$$

were R denotes net available resources that at a given time t splits into maintenance cost of current population -first term on the LHS of the equation- and the investment necessary to increase total inhabitants (second term on the LHS). E_o and E are per capita resources needed for maintenance and for population increase and, of course, $N=N(t)$. In the urban context, population increases naturally with birth/death rates and also with migration. Then E represents an average resource cost between growing a new adult and to "import" one from abroad, net of savings from deaths.

In what follows I discuss the case where resources represent economic wealth given by equation (12), which combined with (14) yields

$$\frac{\partial N}{\partial t} = \frac{A \cdot R_0}{E} N(t)^{\gamma} - \frac{E_0}{E} N(t)^{\rho} \tag{15}$$

The solution of this equation differs, and hence yields different dynamics for cities, depending on the parameters $A \cdot R_0/E$, E_0/E, $\gamma - \rho$ and the initial population N_0. The solution for $\rho = 1$ (Bettencourt et al. 2007, 2008) is:

$$N(t) = \left[\frac{A \cdot R_0}{E_0} + \left(N_0^{1-\gamma} - \frac{A}{E_0}\right) exp\left(-\frac{E_0}{E}(1-\gamma)\, t\right)\right]^{\frac{1}{1-\gamma}} \tag{16}$$

The power law model on wealth of Section 3 justifies a value $\gamma > 1$; essentially thanks to the *Fréchet* distribution condition $\beta > 1$ and to the empirical evidence for economies of scale in infrastructure with *cost parameters* bounded by $0 < \rho \leq 1$ (most likely between 0.8 and 0.9). Considering constant costs per capita in other resources, like housing and individuals' maintenance as shown also by evidence, lead us to assume a sub-linear maintenance cost factor. Therefore, the assumption $\rho = 1$ is plausible to represent an upper bound for this parameter according to the evidence.

As shown by W-B, the solution for $\gamma = 1$ leads to an unbound exponential growth; for $\gamma < 1$ leads to a bounded sigmoidal growth typical of biological systems; for innovation and wealth driven cities with $\gamma > 1$ leads to a faster than exponential and unbounded growth reaching a mathematical singularity with infinite population N_{∞} at a finite amount of time T.

Then if we consider $\rho = 1$, the critical time for growth if $N_0 < N_\infty$, also given by Bettencourt et al. (2007, 2008) is:

$$T = -\frac{E}{(\gamma-1)R_0} ln\left[1 - \frac{R_0}{A} N_0^{1-\gamma}\right] \approx \frac{E}{(\gamma-1)A} N_0^{1-\gamma} \quad ; \tag{17}$$

if $N_0 > N_\infty$ population collapses.

An important note made by W-B is that in equation (15) *T* decreases with initial population (because $1 - \gamma < 1$), which indicates that the growth periods or cycles between N_0 and N_∞ decreases with the city size, what represents an acceleration of cities cycles. To understand this phenomena, Bettencourt et al. 2007) interpret N_0 as the population when a new cycle starts, or when technological, political or other system innovations or adaptation change the dynamics of the city to enter in a new cycle. Also notice that *T* decreases with γ, indicating that (for ρ=1) super-linear wealth and innovation accelerates cities' cycles and thus reducing the time left to produce mayor changes in the system.

For $\gamma > 1$ the wider range of growth with $0 < \rho < \gamma$ can be explored numerically. In this case the solution of equation (13) is an hypergeometric curve with very similar shape as equation (14) and characterized by $\gamma - \rho$, which mesures the *net returns to scale* or the strength of the economic growth: the larger the net returns the steeper the slope of *N(t).* Moreover, according to the model the power that accelerates cities is the *net returns to scale* $(\gamma - \rho)$.

5. FINAL COMMENTS

The above theory emerges from two main axioms. First, all micro decisions in the urban system are governed by agents' rationality that maximize satisfaction from a large set of activities, where each activity yields utility, activities interact among them in the market and agents also interact socially. The second axiom is that agents perceive utility from activities with stochastic variability, which reflects a combination of their idiosyncratic variability -on their perceptions of goods options and environment- and the variability caused by imperfect information and external shocks in the environment where they make choices.

These stochastic optimization behavior is modeled using asymptotic results represented by extreme value distributions: Types I (*Gumbel*) for utilities and profits, and Type II (*Fréchet*) for bids in land markets. Although the *Gumbel* model of micro individual choices is not essential for the main results to hold -it supports that bids distribute independent *Fréchet*- it has the merit of describing, however coarsely, a system wide microeconomic consistency among all (households and firms) individuals' choices and the diversity of the system from which emerges the power laws.

Under plausible conditions of the economy, like non-negative economies of scale in production, the main results of this paper is the emergence of rent and welfare super-linear power laws with population from standard microeconomic assumptions. Following Page (2011), I interpret this results as: *ceteris paribus, total welfare and rents of cities increase with population and with diversity*. Diversity arises from the bids' *Fréchet* parameter β, which yields a natural explanation for the increasing returns to scale. Notably, this parameter emerges from the variance of all agents' variability in all their *Gumbel* model choices, thus encapsulating a complex micro level variability structure of the system that emerges in the land market.

This connection between bids and the scale parameter of the power law is significant because these parameters can be estimated from observed data of households' and firms' locations and the associated observed rents using standard maximum likelihood estimators. However, known estimates assume a *Gumbel* distribution of bids (e.g. Lerman and Kern, 1983; Martínez and Donoso, 2010); it remains for further research to apply this estimation techniques to the *Fréchet* model. Additionally, the transport literature may contribute with their standard studies to test and estimate economies of scale in productive and social activities of the *Gumbel* model.

The critical role of variability in this model implies that the more the variance in the system's aggregate choice process, the lager the expected total welfare and rents. This tells us that more diverse societies are more creative, produce higher number of opportunities for interaction among agents, and the creation of welfare is more likely to occur for a given population. In other words, a society composed by cloned humans with predictable (deterministic) behavior is expected to provide the least welfare among those of the same size, everything else (including culture and scale economies) being equal.

The paper provides a very simple theoretical explanation to the empirical evidence that welfare and rent indicators scale super-linearly with size (population) in worldwide studies. It states that super-linear returns to scale have two sources: the classical economies of scale in economic theory (not clearly relevant according to the evidence) and the effect of diversity in creating better opportunities. From our model, even under non-negative economies of scale, the sole stochastic effect supports super-linear effects.

The essential metaphoric connection between organisms and humans organizations dynamics is simple: both share the assumption of an optimization behavior, the former optimize energy production and its use, humans optimize use of resources (material, economical and time). Nevertheless, there is also an essential difference between them: human systems can grow faster, accelerated not only by scale economies, but also by innovation and opportunities associated with population scale.

ACKNOWLEDGEMENTS

This research was supported by FONDECYT 1110124 and Instituto de Sistemas Complejos de Ingeniería (ICM: P-05- 004-F, CONICYT: FBO16). I also thank G. West, L. Bettencourt and L.G. Mattsson for their motivating and clarifying comments.

REFERENCES

Alonso, W. (1964). **Location and Land Use**, Cambridge, Harvard University Press.

Anas, A., Liu, Y. (2007) A regional economy, land use, and transportation model (Relu-Tran©): formulation, algorithm design and testing. **Journal of Regional Science 47**(3), 416-455.

Anas, A., Hiramatsu, T. (2012). The effect of the price of gasoline on the urban economy: From route choice to general equilibrium. **Transportation Research Part A 46**, 855–873.

Ben-Akiva, M., Lerman, S.R., 1985. **Discrete Choice Analysis: Theory and Application to Travel Demand.** The MIT Press, Cambridge, Massachusetts.

Bettencourt, L.M.A, Lobo, J., Helbing, D., Künhert, C., West, G. (2007). Growth, innovation, scaling, and the pace of life in cities. **PNAS 104**, 17, 7301-7306.

Bettencourt, L.M.A., Lobo, J. and West G.B. (2008). Why are large cities faster? Universal scaling and self-similarity in urban organization and dynamics. **The European Physical Journal B 63,** 285-293.

Black, D., Henderson, V. (1999). A Theory of Urban Growth. **Journal of Political Economy 107(2)**, 252-284.

Cho, S-H., Chen, Z., Yen, S. (2008). Urban growth boundary and housing prices: the case of Knox County, Tennessee. **The review of region Studies 38(1),** 29-44.

Domencic, T.A., McFadden, D. (1975). **Urban Travel Demand: A Behavioral Approach**. North-Holland.

Ellickson, B. (1981). An alternative test of the hedonic theory of housing markets. **Journal of Urban Economics 9** (1), 56–79.

Fisher, R.A., Trippett ,L.H.C. (1928). Limiting forms for the frequency distribution of the largest or smallest member of the sample. **Proc. Cambridge Phil. Soc.** 24, 180-190.

Gabaix, X. (1999). Zipf's law for cities: an explanation. **The Quaterly Journal of Economics 114**, 739-767.

Galambos, J. (1987). **The Asymptotic Theory of Extreme Order Statistics**, Wiley, New York.

Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aleatórie. **Ann. Math. 44**, 423-453.

Henderson, V. (1986). Efficiency of resource usage and city size. **Journal of Urban Economics 19**, 47-70.

Leadbetter, M.R., Lindgren, G., Rootzén, H. (1983). **Extreme and related properties of random sequences and processes**. Springer-Verlag, New York.

Lerman, S.R., Kern, C.R. (1983). Hedonic theory, bid rents and willingness to pay: Some extensions of Ellikson's results. **Journal of Urban Economics 12(3)**, 358-363.

Mattsson, L-G., Weibull, J., Lindberg, P. O. (2011). Extreme values, invariances and choice probabilities. Manuscript presented at the Kuhmo Nectar Conference on Transport Economics, Stockholm.

Martínez, F., Donoso, P. (2010). The MUSSA II Land Use Auction Equilibrium Model. In F. Pagliara, J. Preston, D. Simmonds (eds.), **Residential Location Choice: Models and Applications**. Advances in Spatial Science, Springe.

Martínez, F., Hurtubia, R. (2006). Dynamic model for the simulation of equilibrium states in the land use market. **Networks and Spatial Economics 6,** 55-73.

Ortúzar, J. de D., Willumsen, L.G. (2001). **Modelling Transport**. Chichester: John Wiley and Sons.

Page, S.E. (2011). **Diversity and Complexity.** Princeton University Press, New Jersey.

Rosen, S. (1974). Hedonic prices and implicit markets: product differentiation in pure competition. **Journal of Political Economy 82**, 1, 34-55.

Von Thünen, J.H. (1863). See **Von Thünen Isolated State**, (ed.) Peter Hall, Pergamon Press, London. (1966) .

West, G.B. (1999). The origin of universal scaling law in biology. **Physica A 263,** 104-113.

West, G.B., Brown, J.H., Enquist, B.J. (1999). The fourth dimension of life: Fractal geometry and allometric scaling of organism. **Science 284**, 1667-1679.

West, G.B., Brown, J.H., Enquist, B.J. (2001). A general model of ontogenetic growth. **Nature 413,** 628-631.