ON THE DOWNS-THOMSON PARADOX UNDER TRANSIT DISPATCHING AND PRICING SCHEMES

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ABSTRACT

It is observed theoretically and empirically that the Downs-Thomson Paradox may occur in the sense that capacity expansion produces counterproductive effect on the overall network performance. This happens where the highway is in parallel of a transit line on the same corridor, and the shifting of traffic volume from transit system could make the highway more congested while transit service level also decreases due to shrinking revenue. The purpose of this paper is to investigate the occurrence of the Downs-Thomson Paradox considering monopoly transit dispatching and pricing schemes. We analyze the relationship between volume change and transit polices when highway capacity is expanded. Furthermore, the impact of capacity change on transit policies is presented. The conditions for the occurrence of Downs-Thomson Paradox are the most important findings. The results obtained will be demonstrated with numerical examples.

Keywords: Downs-Thomson Paradox, Highway Capacity Expansion, Transit Dispatching Policy, Transit Fare Price

1. INTRODUCTION

1.1. Motivation and background

Modern commuters worldwide have long been bearing about the annoying traffic congestion on the road. Traffic congestion has become one of the most important concerns in urban planning. Many countries mostly depend on supply-side policies to mitigate urban transportation congestion, such as through expanding network capacity and improving traffic management. However, the arguments towards short-sighted capacity expansion are explosive after observing its implementation for about a century.

Downs (1962) first claimed that in response to capacity expansion, the peak-hour traffic congestion will soon rises to meet the maximum capacity from three immediate effects. The most significant one is the mode shifting from public transit to the newly added highway capacity. As consequences, no matter how many new superroads are built, auto-mobile commuters can only move at scrawl during the morning and evening rush hours.

The whole picture is as follows: At first, higher capacity reduces automobile travel time on highway, and some of the informative transit passengers would give up their original choices and come to highway immediately. As a result, the loss of transit patronage means decrease in ticket revenue, and the monopoly transit operator would have to lower down the service frequency in order to curtail the operation costs. This feedback effect would drag the equilibrium further to a higher automobile volume and lower welfare level for the overall system. This well-known phenomenon that the generalized travel costs of both modes increase after improvement of the highway capacity due to the effect of the volume shifting is called “Downs-Thomson Paradox”. Such a phenomenon is a significant argument for people who are defending urban pricing for private cars and consequently investment in highway capacity systems, in order to increase journey speeds.

1.2. Literature review

Since Anthony Downs (1962) first introduced the paradox mechanism in the paper “The Law of Peak-Hour Expressway Congestion”, much attention was paid to this problem. Thomson (1977) proposed the paradox phenomenon through empirical studies based on real data collected from thirty world's great metropolitan cities. His study depicted a fair picture of the metropolitan transport problem. Mogridge et al. (1987) suggest that it may occur by allocating even more space to roads when roads are a less efficient carrier of the flow of traffic. Holden (1989) suggests that it may occur in a city like London, where a significant fraction of peak traffic is carried on an extensive rail network. These statements are mainly qualitative, based on intuition and experience.

To understand the economics of this problem, Arnott and Small (1994) provides a synthesis analysis of the paradox occurrence employing economic concepts of externalities and illustrative examples. They indicated that the resolution of the paradox would not only clarify the economics of traffic congestion, but also point ways in which the congestion problem can be solved with clever application of standard pricing tools of economics. Ding and Song (2008) also addressed the economics of the problem, and further proposed the economic resolution through the tool of social marginal cost pricing. Under appropriate pricing schemes, the system could reach the social optimal state by internalizing the travel externality and discouraging commuters to use the private cars.

Abraham and Hunt (2001) analyzed the process of paradox occurrence in detail and showed the specific aspects of the mode split. They proposed a method to test whether a system is in a state where the Downs-Thomson Paradox may occur using the transit service level function. Denant-Boèmont and Hammiche (2009) built an experiment to observe the Downs-Thomson paradox empirically in the laboratory: an increase in highway capacity causes shifts from rail to highway and, at the end, increases total travel costs. Micheal Bell (2011) revisited the Downs-Thomson Paradox from the perspective of road use charging and inter-modal equilibrium and evaluated the efficiency of the prevailing congestion charge theory.

Literature from the field of the so called “two-mode problem” (Ahn, 2009; Yang and Huang, 2005; Yang and Woo, 2000; Yang et al., 2009; Small, 1992; Small et al., 2007; Daganzo, 2012; Li, 2012) provides theoretical basis for studies on the paradox. Arnott and Yan (2000) discussed the paradoxical effect of highway capacity expansion considering the transit strategy while the highway users are underpriced. They pointed out that in deciding on transit or highway policies, the authority should take the effects of substitution into account. The transit operation strategy is indispensable to cause the problem. According to the results of Reinhold (2008), when a transit line operator is faced with low demand, he tends to cut expenses instead of improving level-of-service. Bar-Yosef et al. (2012) formalized the passengers’ decisions in modal choice to show how the emergence of a vicious cycle depends on the characteristics of potential passengers. Kraus (2003, 2012) formulated the optimal mass transit problem where different highway tolling schemes are incorporated. Light (2009) developed an analytical framework to characterize the optimal toll and capacity policies which can be used in the study of the paradoxical effect.

1.3. Contributions of this paper

In view of the foregoing discussion, this paper proposed analytical models for investigating the relationship between the transit system design issues and the Downs-Thomson effect in a simple network with auto and transit interactions. (i.e. both fare and frequency)

While considerable progress has been made in understanding the economics of the problem, the existing literature is still unable to provide precise quantitative conditions under which paradox will occur, especially considering transit dispatching and pricing schemes. The main purpose of this paper is to find practical conditions to identify whether the system is in a state where the Downs-Thomson Paradox would occur. To this end, the specific impacts of changes of highway or transit decisions on the mode shifting effects are analyzed under different transit management scenarios.

The paper is organized as follows. In the next section, we introduce the basic model and define the main problem. In Section 3, we analyze the transit dispatching and pricing schemes and find out the necessary and sufficient conditions for the occurrence of Downs-Thomson Paradox under different scenarios. In Section 4, a numerical example is given to illustrate the essential merits of the proposed models. Conclusions are given in Section 5.

2. MODEL FORMULATION

2.1. Problem settings

We consider a simple single O–D pair model to present a corridor with a congested highway running in parallel with exclusive transit line, linking the residential area and the central business district, as shown in Figure 1.

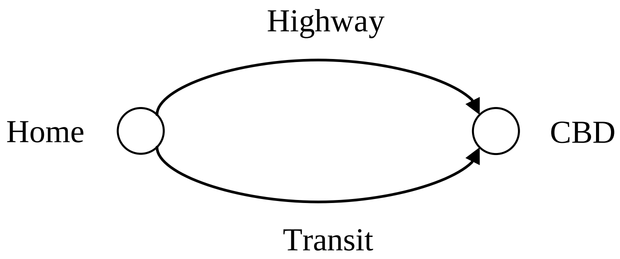

Figure 1 - A simple two-mode network

The main assumptions are listed to facilitate the analysis of this paper:
(a) Fixed total demand.
The total demand is assumed to be fixed and denoted by d. Based on this assumption, external demand elasticity is ignored and the analysis would focus on the internal competition effect between the two modes.

(b) Homogeneous commuters with identical value of time.
Commuters with the same value of time, denoted by β, can choose either transit or highway to reach the CBD every morning.

(c) Congested highway.
Suppose that the travel time on the highway, denoted by $t_a = t_a(v,c)$ where v is the number of commuters using c units of highway capacity. The travel time function is assumed to satisfy $\partial t_a(v,c)/\partial v > 0,\ \partial t_a(v,c)/\partial c > 0,\ \partial^2 t_a(v,c)/\partial v^2 > 0$ and $\partial^2 t_a(v,c)/\partial v \partial c < 0$, which are widely used in practice.

(d) Underutilized transit.
We assume that the holding capacity of each train is large enough to carry all the commuters waiting on the platform, no matter how many people are aligning on the platform, every commuter can surely get on-board as soon as a train comes. After boarding, the in-vehicle travel time is assumed to be a constant, denoted by t_t, and is larger than the free-flow travel time on the highway, $t_t > t_a(0,c)$.

(e) Positive transit fare and maximum frequency limitation.
Denote (τ, f) as the transit operating decision, where $\tau > 0$ be the positive uniform fare price, and $f > 0$ the train frequency. We assume that transit is operating under fixed schedule, and every single trip has identical and fixed running time. Existing trains have the fixed and

identical holding capacity, given by a constant $h > 0$. The maximum frequency given by the safety margin is assumed to be a constant $f < \bar{f}$.

(f) Monopoly transit operator and its convex operation cost.

Monopoly transit operator makes decisions of fare and frequency according to the transit patronage under the principle of zero-profit. The operator collects money from the fare revenue while pays for the transit operation cost, denoted by $k = k(f)$. Here we ignore the other cost components, such as capital costs for fleet of cars and terminals, or the construction costs of trackage (Jansson ,1980; Kraus, 2003), and assume the operation cost only depends on the frequency of runs. It is the summation of a positive fixed cost k^0 and variable cost which is increasing with the frequency. It is an increasing function of frequency and assumed to be convex as shown in Figure 2, follows that $k(0) = k^0 > 0,\ k' > 0,\ k'' > 0$. This assumption could well reflect the reality that even the transit operator stops the service so that the frequency is zero, there will still be a fixed cost as a result of the long-term fixed expenditures that couldn't be prohibited. Besides, when the frequency becomes very large, the operation cost will go extremely high because of technique limitations and safety precautions. And it implies that we ignore the other costs associated with the transit operation, such as infrastructure construction cost, labor cost, etc. The profit of transit operator is given by $\pi = \tau \cdot (d - v) - k(f)$.

(g) Travel by transit: convex waiting time.

In addition to the monetary cost, each transit user's travel time cost consists of two parts, waiting time and in-vehicle travel time, denoted by w and t_t (both are positive), respectively. In reality, most unscheduled commuters must wait at the platform until the train comes, thus average individual waiting time is a decreasing function of the train frequency, i.e., $w = w(f)$. Following the assumption in the Ahn (2009), waiting time function is convex and decreases with frequency, follows $w' < 0,\ w'' > 0$. When frequency is approaching the maximum limitation, the waiting time would reach to the lower bound, denoted by $w(\bar{f})$, as shown in Figure 2.

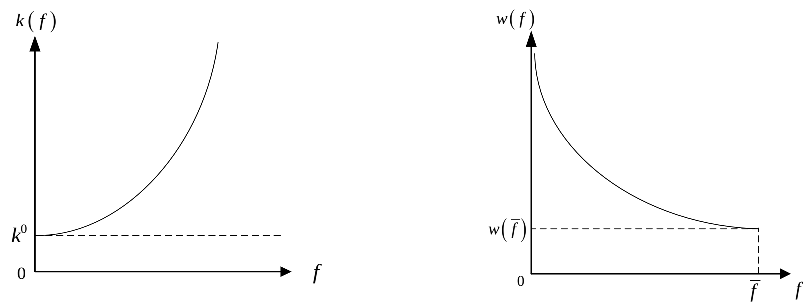

Figure 2 - The Properties of Operation Cost and Waiting Time Function

(h) No monetary charge by auto with only travel time cost.

Commuters who choose automobile can use the highway free of charge, and therefore travel time is the only cost.

(i) Continuity of highway capacity.
Highway capacity, denoted by c, is subject to adjust and assumed to be continuous. The assumption of continuous highway capacity is somewhat unrealistic given that highway lanes must be provided in discrete increments but this assumption is standard in the theoretical literature which covers highway capacity provision (Light, 2009). We set the current capacity level as the lower-bound of highway capacity, follows $c \geq c_0$, where c_0 denotes the current capacity level and it is assumed that $t_a(d,c) > w(\bar{f}) + t_t$ in order to make sure that the volume will never go to zero or d. To focus on the analysis of transit dispatching and pricing schemes, we wouldn't spare efforts on analyzing the economic objective or decision-making process of the highway operator, and as a consequent, neither the highway investment nor operation cost is taken into account.

Now the elements of the two-mode system could be presented in Table I:

Table I –Description of the two-mode system

	Service level	Volume	Commuter's cost	Decision constraints
Transit	(τ, f)	$d-v$	$w(f),\ t_t,\ \tau$	$\pi = \tau \cdot (d-v) - k(f)$ $(d-v)/H \leq f \leq \bar{f}$
Auto	c	v	$t_a(v,c)$	$c \geq c_0$

The above settings allow us to express the generalized travel disutility, which combines the value of travel time and the total monetary travel cost. Let $U_i,\ i = t,\ a$ denote the generalized travel disutility of an individual commuter where t represents transit and a represents automobile, and it is the summation of monetary cost and the time spent on the travel. For a transit commuter, the total time spent involves both the constant in-vehicle time and the waiting time at the platform, and the his total disutility can then be given by

$$U_t = \tau + \beta \cdot \left[w(f) + t_t \right].$$

On the other hand, an automobile commuter have only the travel time cost since there assumed to be no congestion charge on the highway, and his generalized travel disutility is as follows,

$$U_a = \beta \cdot t_a(v,c).$$

2.2. Deterministic mode choice and traffic equilibrium

Deterministic user equilibrium is achieved when no commuter can reduce his or her travel cost by changing to an alternative travel mode, given the choices of other commuters. That is, a commuter will compare the generalized travel disutilities of both modes, and select the one with least cost to him or herself, taking others' decisions as given. The equilibrium volume will thus be determined under deterministic user equilibrium principle: At equilibrium, the generalized disutility of both modes is identical, follows that

$$U_t = U_a \Leftrightarrow \tau + \beta \cdot \left[w(f) + t_t \right] = \beta \cdot t_a(v,c). \tag{1}$$

Proposition 1. *For any given* $f \in \left(0, \bar{f}\right]$, $c \in (0, \bar{c}]$ *and* $\tau \in \left(0, \beta \cdot \left(t_a(d,c) - w\left(\bar{f}\right) - t_t\right)\right]$, *there exists a unique equilibrium volume* v *that solves Equation (1).*

Proof. Let $\tau(v) = \beta \cdot \left[t_a(v,c) - w(f) - t_t\right]$, and it suffices to show that there exists unique v such that $\tau(v) = \tau$ for all $\tau \in \left(0, \beta \cdot \left(t_a(d,c) - w\left(\bar{f}\right) - t_t\right)\right]$. Note that $\tau\left(t_a^{-1}\left(w(f) + t_t, c\right)\right) = 0$, $\tau(d) > 0$, and $\tau'(v) > 0$ for all $v \in \left(t_a^{-1}\left(w(f) + t_t, c\right), d\right)$. Therefore, by the continuity of $\tau(v)$ and the intermediate value theorem, there exists a unique v such that that $\tau(v) = \tau$ for all $\tau \in \left(0, \beta \cdot \left(t_a(d,c) - w\left(\bar{f}\right) - t_t\right)\right]$.■

2.3. Downs-Thomson paradox

The Downs-Thomson Paradox occurs when the generalized travel costs of both modes increase after improvement of the highway capacity due to the effect of the volume shifting. As stated by Downs (1962), the whole picture is as follows: At first, higher capacity reduces automobile travel time on highway, and some of the informative transit passengers would give up their original choices and come to highway immediately. As a result, the loss of transit patronage means decrease in ticket revenue, and the monopoly transit operator would have to lower down the service frequency in order to curtail the operation costs. This feedback effect would drag the equilibrium further to a higher automobile volume and lower welfare level for the overall system.

Mathematically, the direct effect of capacity expansion is the partial derivative of generalized travel cost with respect to the highway capacity:

$$\frac{\partial U_a}{\partial c} \cdot \Delta c < 0.$$

Following Abraham (2001), to analyze the total effect of highway capacity improvement, we should evaluate the total derivative of the generalized travel disutilities of both modes at the potential capacity level. Note that the change direction of the total disutilities of both modes should be identical because of the equilibrium condition. Now we are ready to define the paradox condition:

Definition Downs-Thomson Paradox (D-T Paradox) is said to occur at the point $c = c^*$, if the total derivatives of the generalized travel disutilities at equilibrium with respect to highway capacity is positive evaluating at $c = c^*$:

$$\frac{dU_a}{dc} = \left.\frac{dU_t}{dc}\right|_{c=c^*} > 0. \tag{2}$$

3. PARADOX CONDITION UNDER DIFFERENT TRANSIT SCHEMES

In this section we analyze the possibility of paradox and the region where it occurs under the following three scenarios when (1) monopoly transit operator can freely tune the frequency while fare price is regulated by political authority, (2) transit fare price can be adjusted while service frequency is predetermined, and (3) both frequency and fare price are subject to changes.

3.1. Transit dispatching scheme

In this subsection, we analyze the scenario that the transit operator could freely adjust the dispatching scheme on service frequency according to the change of highway capacity, while set the original fare price as fixed. In view of political concerns, it is necessary for a government sets up a standard pricing scheme for the public transit service that the monopoly operator couldn't violate in the short-run. In fact, in most cities worldwide, public transit service has a relatively fixed framework of price no matter it is running by monopoly operator or not.

As the direct effect of highway capacity improvement, there will be a decrease in transit patronage, and the original equilibrium will fail and transit operator can't maintain the break-even point. To response, the operator should come up with corresponding frequency based on the rule of user's equilibrium and his zero-profit objective. The adjustment in transit dispatching scheme will lead the system to a new equilibrium point, which is determined by:

$$\begin{cases} \tau_0 + \beta \cdot \left[w(f) + t_t \right] = \beta \cdot t_a(v, c) \\ \tau_0 \cdot (d - v) - k(f) = 0 \end{cases}$$

where τ_0 is the fixed fare price and $f = f(v)$ is the transit operator's dispatching scheme.

Proposition 2. *At any equilibrium point, the sufficient and necessary condition for the occurrence of D-T Paradox under fixed transit fare is given by* $t_v' > -\tau_0 \cdot w'/k'$*, where* t_v' *is the first-order derivative of highway travel time with respect to* v*,* w' *and* k' *are the derivative of transit waiting time and operation cost with respect to* f*, respectively.*
Moreover, if the D-T Paradox occurs when $c = c_0$*, it will occur in the interval* $[c_0, \bar{c}]$*, where* $\bar{c}$ *is the solution to* $t_v' + \tau_0 \cdot w'/k' = 0$*.*

Proof. According to the zero-profit constraint, transit frequency should decrease with v, and the marginal effect on frequency of the volume change is such that

$$f_v' = -\frac{\tau_0}{k'},$$

Where f_v' is the derivative of frequency with respect to v. As assumption (f) describes, $k' > 0$.

The total change in volume resulted from the change of highway capacity can be captured by analyzing the first-order derivative of equilibrium condition:

$$\frac{dU_t}{dc}-\frac{dU_a}{dc}=0$$
$$\Leftrightarrow \beta\cdot w'f_v'\cdot dv/dc-\beta\cdot\left(t_c'+t_v'\cdot dv/dc\right)=0$$
$$\Leftrightarrow \frac{dv}{dc}=-\frac{t_c'}{t_v'-w'f_v'}$$

Now we are ready to derive the paradox condition. According to the definition given by Equation (2), the paradox will occur when the total change in travel distility with respect to highway capacity is positive:

$$\frac{dU_a}{dc}>0$$
$$\Leftrightarrow \beta\cdot w'f_v'\cdot dv/dc>0$$
$$\Leftrightarrow t_v'>-\tau_0\cdot w'/k'$$

Let $I_1=t_v'+\tau_0\cdot w'/k'$, the argument for the second part of Proposition 2 is immediate by noting that I_1 is strictly decreasing with c:

$$I_1'=\frac{\partial^2 t_a\left(v,c\right)}{\partial v\partial c}+\tau_0\cdot\frac{w'f_v'k'\cdot dv/dc-k'f_v'w'\cdot dv/dc}{k'^2}<0\,.\blacksquare$$

Proposition 2 implies that under a fixed transit fare price, the D-T Paradox will occur when the negative externality on transit overwhelms the positive effect on the highway resulted due to the capacity expansion. This gives a method to examine the occurrence of D-T Paradox by checking the sign of index $I_1=t_v'+\tau_0\cdot w'/k'$ at the equilibrium points. Furthermore, it is found that once the condition is satisfied at the initial point, it will continue to be active until the capacity reaches the boundary condition that I_1 hits zero. This is to say, the D-T Paradox will occur in the interval $\left[c_0,\,\bar{c}\right]$, and the upper-bound $\bar{c}$ is determined by the zero point of index I_1.

3.2. Transit pricing scheme

In this subsection, we analyze the scenario that the transit operator has the freedom on determining the fare price according to the change of highway capacity, while the service frequency has to be fixed at a certain level. This scenario is also drawn from the reality that for the sake of social welfare, some local governments legislate on public transit to regulate the service quality of the monopoly public transit operator.

Similar to the previous scenario, the original equilibrium will fail and there will be a decrease in transit patronage as the direct effect of highway capacity improvement. The adjustment in transit pricing scheme is determined by:

$$\begin{cases}\tau+\beta\cdot\left[w\left(f_0\right)+t_t\right]=\beta\cdot t_a\left(v,c\right)\\ \tau\cdot\left(d-v\right)-k\left(f_0\right)=0\end{cases}$$

where f_0 is the fixed frequency and $\tau = \tau(v)$ is the pricing scheme.

Proposition 3. *At any equilibrium point, the sufficient and necessary condition for the occurrence of D-T Paradox under fixed transit frequency is given by* $\tau < \beta(d-v)\cdot t'_v$*. Moreover, if the D-T Paradox occurs when* $c = c_0$*, it will occur in the interval* $[c_0, \tilde{c}]$*, such that* $\tilde{c}$ *is the solution to* $\tau - \beta(d-v)\cdot t'_v = 0$*.*

Proof. According to the zero-profit constraint, transit fare should increase with v, and the marginal effect on fare of the volume change is such that

$$\tau'_v = \frac{\tau}{d-v},$$

where τ'_v is the derivative of fare with respect to v.

Similar to the proof of Proposition 2, the total change in volume resulted from the change of highway capacity is given by:

$$\frac{dv}{dc} = \frac{\beta t'_c}{\tau'_v - \beta t'_v}$$

Now we are ready to derive the paradox condition is given by:

$$\tau < \beta(d-v)\cdot t'_v.$$

For the second part of Proposition 3, let $I_2 = \beta(d-v)\cdot t'_v - \tau$, and assume that the paradox condition is satisfied at the initial point, such that $I_2\big|_{c=c_0} = \beta(d-v)\cdot t'_v - \tau > 0$. The argument is immediate by noting that I_2 is strictly decreasing with c:

$$I'_2 = \beta(d-v)\cdot\frac{\partial^2 t_a(v,c)}{\partial v \partial c} - \beta t'_v \cdot \frac{dv}{dc} - \tau'_v \cdot \frac{dv}{dc} < 0. \blacksquare$$

Proposition 3 implies that under a fixed transit service frequency, the D-T Paradox will occur when the transit fare price is higher than the total saving by using automobile. This gives a method to examine the occurrence of D-T Paradox by checking the sign of index $I_2 = \beta(d-v)\cdot t'_v - \tau$ at the equilibrium points. Similar to the previous scenario, it is found that once the condition is satisfied at the initial point, it will continue to be active until the capacity reaches the boundary condition that I_2 hits zero. This is to say, D-T Paradox will occur in the interval $[c_0, \tilde{c}]$, where $\tilde{c}$ is the solution to $\beta(d-v)\cdot t'_v - \tau = 0$.

3.3. Multi-scheme

In this subsection, a more general scenario will be considered where no regulations are imposed on either transit frequency or fare price, and the transit operator has the entire freedom on choosing both the dispatching and the pricing schemes. The problem for the transit operator is reduced to:

$$\beta\cdot\left[t_a(v/c) - w(f) - t_t\right]\cdot(d-v) - k(f) = 0. \tag{3}$$

Please note that even though no exogenous constraints are imposed in this scenario, it does not necessarily mean that the transit operator can arbitrarily choose frequency and fare. In fact, the decision is subject to the following endogenous constraints resulted from the basic assumptions: (1) the fixed cost of transit operation, (2) positive fare price, and (3) maximum frequency for safety margin. As consequences, the constraints are given by:

$$\begin{aligned} &\tau \in \left(\frac{k(0)}{d-v}, \infty \right) \\ &f \in \left(w^{-1}\left(t_a(d, c_0) - t_t \right), \bar{f} \right) \end{aligned} \tag{4}$$

Now we come to the analysis of paradox condition. As previously mentioned, the impact of highway capacity expansion is realized in two steps. The direct effect refers to the reduction in automobile travel time on highway, and so that some of the informative transit passengers would come to highway immediately. While indirectly, the changes in transit frequency and fare price further increase automobile volume and lower welfare level for the overall system. To find out the condition of paradox, we capture the total effect of capacity expansion through equilibrium analysis and conclude in the following proposition.

Proposition 4. *At any equilibrium point, the sufficient and necessary condition for the occurrence of D-T Paradox is given by*

$$0 < \beta w' f'(d-v) + \tau + k' f' < \beta t_v'(d-v). \tag{5}$$

Proof. The total change in volume resulted from the change of highway capacity can be captured by analyzing the first-order derivative of Equation (3):

$$\begin{aligned} &\beta \cdot (d-v) \cdot \left(t_c' + t_v' \frac{dv}{dc} - w' f' \frac{dv}{dc} \right) - \beta \cdot \left[t_a(v/c) - w(f) - t_t \right] \cdot \frac{dv}{dc} - k' f' \frac{dv}{dc} = 0 \\ \Leftrightarrow\ & \frac{dv}{dc} = -\beta t_c'(d-v) \left[\beta \left(t_v' - w' f' \right)(d-v) - \beta \left(t_a - w - t_t \right) - k' f' \right]^{-1} \end{aligned}$$

According to the definition given by Equation (2), the paradox will occur when the total change in travel distility with respect to highway capacity is positive:

$$\begin{aligned} &\frac{dU_a}{dc} > 0 \\ \Leftrightarrow\ & \begin{cases} \beta w' f'(d-v) + \beta \left(t_a - w - t_t \right) + k' f' > 0 \\ \beta \left(t_v' - w' f' \right)(d-v) - \beta \left(t_a - w - t_t \right) - k' f' > 0 \end{cases} \\ \Leftrightarrow\ & 0 < \beta w' f'(d-v) + \tau + k' f' < \beta t_v'(d-v). \blacksquare \end{aligned}$$

This condition includes two parts. In the first part, $k'f'$ is the marginal operation cost, and $\beta w' f'(d-v)$ is the marginal total waiting time of all the transit passengers, while τ is the transit fare price. In the second part, $\beta t_v'(d-v)$ is the total saving by transit. To sum up, at the points that the transit fare price is higher than the sum of the marginal operation cost and the marginal total waiting cost, while less than the sum of the marginal operation cost, the marginal total waiting cost and the total saving by transit, the Downs-Thomson Paradox would occur.

4. A NUMERICAL EXAMPLE

To facilitate the presentation of the essential ideas, we employ an example in this section to illustrate the numerical application of the proposed models. Consider the two-mode network described in Section 2 with the following parameters:

The total demand in an hour is fixed, $d = 500$ person. The transit operation cost function takes the form that $k(f) = k^0 + k^1 \cdot f^\gamma$. It is the summation of the fixed cost $k^0 = 200$ HK\$ and variable part with parameters $k^1 = 10$ HK\$/train, $\gamma = 2$. The waiting time function takes the form that $w(f) = 1/(2f)$. The in-vehicle travel time on transit is fixed, $t_t = 45$ min. The holding capacity of each train is identical, $h = 2000$ person. The maximum frequency is $n = 20$. The highway travel time function takes the form of BPR function that $t_a(v/c) = t_0 \cdot \left[1 + t_1 (v/c)^\alpha\right]$, where $t_0 = 40$ min, $t_1 = 0.15$, $\alpha = 4$. The original highway capacity is given by $c_0 = 200$ person.

With the assumed parameters, for any given transit frequency, fare price and highway capacity, the volume and corresponding travel disutility can be obtained by solving Equation (1). By varying the corresponding variables, we can obtain the equilibrium travel disutility contours in the two-dimensional space.

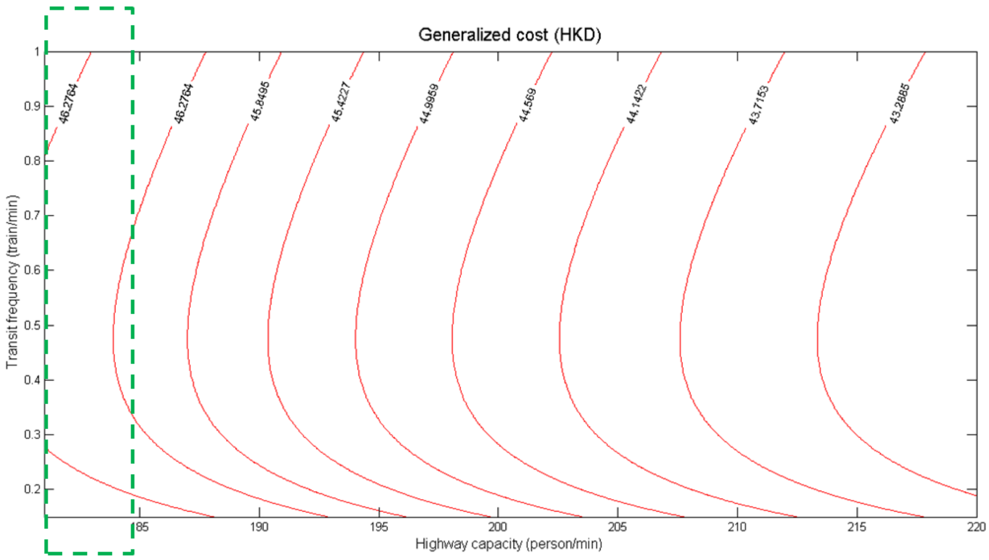

Figure 3 - Total disutility contour under different frequency (fixed fare price).

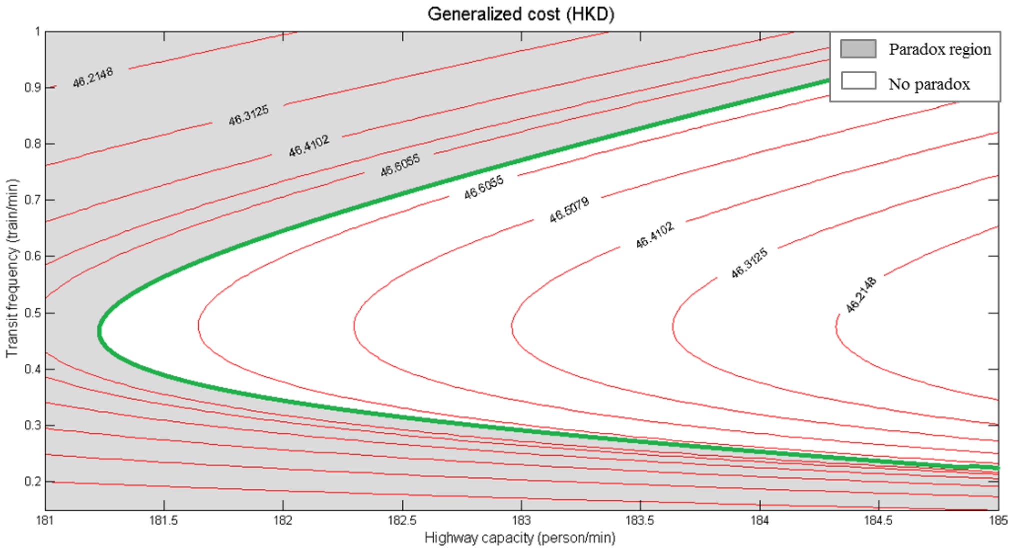

Figure 4 - Paradox region under different frequency levels (fixed fare price).

Figure 3 displays the total travel disutility contours at each equilibrium point under different transit frequency levels with fixed a transit fare price. In this case, we set the transit fare price fixed at $\tau_0 = 0.6\ \text{HKD}$, while the transit frequency and highway capacity vary in the region $f:\ 0.1 \sim 1\ \text{train/min},\ c:\ 181 \sim 300\ \text{person/min}$. We can easily find from the travel disutility contours that the paradox only occurs when the highway capacity is relatively low, which is within the area indicated by the green dashed rectangular.

Figure 4 displays the paradox region in grey, which is the enlarged picture from the marked area in Figure 3. It follows that for a given original capacity, if the original frequency is feasible and in a state of D-T Paradox, it will remain in that state until capacity exceeds the interval given in the previous section.

Table II - Travel disutility and sign of index I_1 (scenario I)

Capacity (person/min)	182		184		186	
Frequency (train/min)	Disutility (HKD)	Index sign	Disutility (HKD)	Index sign	Disutility (HKD)	Index sign
0.3	**46.8**	**+**	**46.5**	**-**	**46.3**	**-**
0.6	**46.6**	**-**	**46.3**	**-**	**46.1**	**-**
0.9	**46.3**	**+**	**46.6**	**+**	**45.7**	**-**

Table II shows the travel disutility and the sign of corresponding index given in the previous section evaluated at the specific equilibrium points. We can see from the table that the index is positive where the disutility is increasing with the capacity while becomes negative in the decreasing region, which is consistent with the statement before.

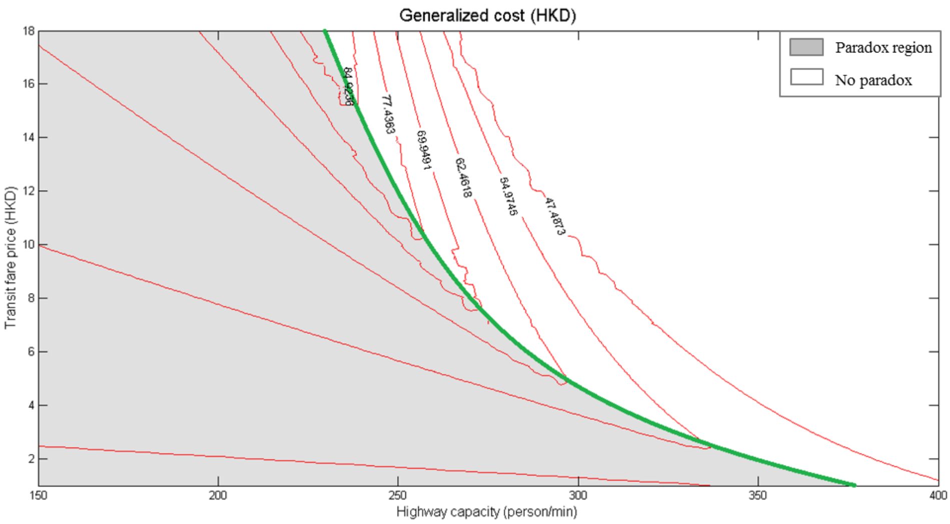

Figure 5 - Paradox region under different fare prices (fixed frequency).

Figure 5 displays the total travel disutility contours and shows the paradox region in grey at each equilibrium point under different transit fare prices with a fixed transit frequency level. In this case, we set the transit frequency fixed at $f_0 = 0.2$ train/min, while the transit fare price and highway capacity vary in the region $\tau : 1 \sim 18$ HKD, $c : 150 \sim 400$ person/min. We can easily find from the travel disutility contours that the paradox only occurs when the highway capacity is relatively low. Similar to the precious scenario, for a given original capacity, if the original frequency is feasible and in a state of D-T Paradox, it will remain in that state until capacity exceeds the interval given in the previous section.

Table III - Travel disutility and sign of index I_2 (scenario II)

Capacity (person/min)	300		350		400	
Fare (HKD)	Disutility (HKD)	Index sign	Disutility (HKD)	Index sign	Disutility (HKD)	Index sign
1.8	**48.7**	**+**	**52.3**	**+**	**45.8**	**-**
2.2	**49.3**	**+**	**52.9**	**-**	**44.1**	**-**
2.8	**46.9**	**+**	**50.2**	**-**	**42.0**	**-**

Table III shows the travel disutility and the sign of corresponding index given in the previous section evaluated at the specific equilibrium points. We can see from the table that the index is positive where the disutility is increasing with the capacity while becomes negative in the decreasing region, which is consistent with the statement before.

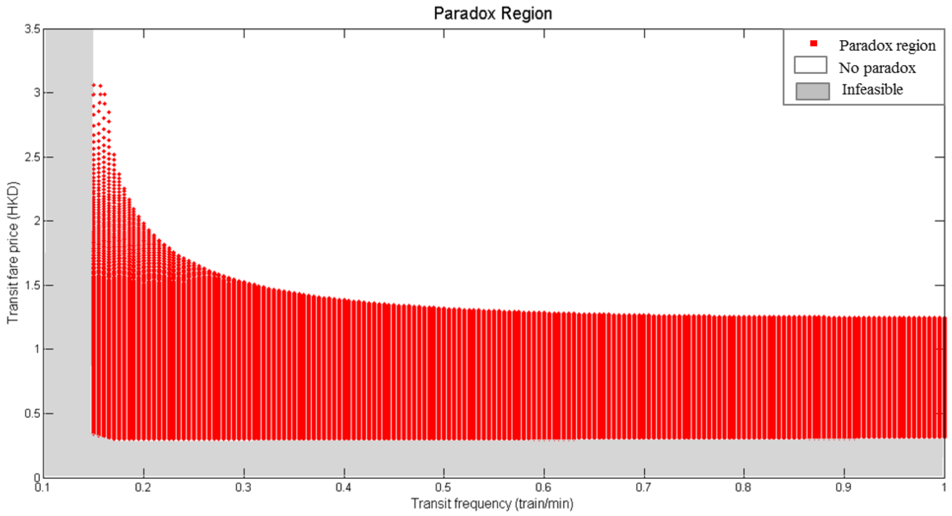

Figure 6 - Paradox region under multi-schemes.

Figure 6 displays the paradox region in red evaluated at each equilibrium point under different transit fare prices and frequency levels. In this case, the regions of transit fare price, frequency, highway capacity are given by $\tau: 0 \sim 3\ \mathrm{HKD}$, $f: 0.1\text{~}1\ \mathrm{train/min}$, $c: 150 \sim 400\ \mathrm{person/min}$, respectively. We can easily find that the paradox only occurs when the transit fare is relatively low, and the paradox region of higher frequency is narrower than that of the low frequency, which implies that with higher frequency level, the range for a safety fare price is much broader. The area in grey is the infeasible area corresponds to the constraints given by Equation (4).

Table IV - Travel disutility and paradox condition (scenario III)

Capacity (person/min)	Frequency, fare (train/min, HKD)	Disutility (HKD)	Condition
200	0.275, 0.706	47.5	√
300	0.253, 1.512	49.7	√
400	0.356, 2.150	56.6	×

Table IV shows the travel disutility and whether it satisfies the corresponding paradox condition given by Equation (5) evaluated at the specific equilibrium points. We can see from the table that the condition is satisfied in the paradox region while is violated in other regions, which is consistent with the statement before.

5. CONCLUSION

This paper investigated the occurrence of the Downs-Thomson Paradox considering monopoly transit dispatching and pricing schemes and analysed the relationship between volume change and transit polices when highway capacity is expanded. Furthermore, the impact of capacity change on transit policies is presented. The conditions for the occurrence of Downs-Thomson Paradox are the most essential findings:

In the scenario with dependent frequency and fixed fare price, an index is given to check the occurrence of Downs-Thomson Paradox. When the sign of the index is positive, an increase in capacity will result in the occurrence of D-T Paradox in a domain, and it implies that negative externality on transit overwhelms the positive effect on the highway.

In the scenario with dependent fare price and fixed frequency, another index is given. Similar to the previous scenario, when the sign of the index is positive, D-T Paradox will occur in a domain, and it implies that when the transit fare price is higher than the total saving by using automobile.

In the scenario with free fare price and frequency, the condition is given by an inequity system, which implies that at the points that the transit fare price is higher than the sum of the marginal operation cost and the marginal total waiting cost, while less than the sum of the marginal operation cost, the marginal total waiting cost and the total saving by transit, the Downs-Thomson Paradox would occur.

The numerical example in the last section illustrates the analytical results and identifies the specific paradox domains.

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