THE DYNAMIC MIXED STRATEGY USER EQUILIBRIUM PROBLEM

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ABSTRACT

Traffic assignment models have been well studied over the years and new methods of data sharing have been evolved. As a result, there has been a great interest in developing stochastic and mixed strategy models. Still, the equilibrium conditions in most traffic assignment models are based on pure strategies. In the mixed strategy approach, each user chooses a probability distribution over his or her pure strategies (where a pure strategy defines a path). The mixed strategy approach relaxes the assumption of homogeneity among network users. Accordingly, a Mixed-Strategy User-Equilibrium model for general networks was developed. However, the mixed strategy path-choices models proposed so far, assumed static conditions. Therefore, this paper introduces a new model in which network users employ mixed strategies under dynamic conditions. The paper investigates the existence and uniqueness conditions for the solution of the dynamic mixed strategy user equilibrium problem. Model application is illustrated for simple network examples.

Keywords: Dynamic Traffic Assignment; User Equilibrium; Path-Choices; Decision Modeling; Mixed Strategies

INTRODUCTION

The problem of predicting flow patterns in transportation networks, also known as the Traffic Assignment (TA) problem, plays a major role in the transportation engineering area (Kacharoo and Ozbay, 1999). The network flow pattern is affected by three types of properties: Physical properties of the network (topology and links' performance functions); Demand (the number of network users between each pair of origin-destination nodes in the network); and decisions of the network's users, manifested through their path choices.
This paper focuses on the following problem: given the physical properties of the network and the demand, and given that the travelers through the network act in a selfish manner, find a network's flow pattern varying over time, dynamically.

Traditionally, the TA problem was modeled by static assignment models (User Equilibrium models), which are descriptive methods for predicting network flows and travel costs under the assumption of non-cooperative users. These models were formulated to be consistent with Wardrop's (1952) first principle, which is based on two main assumptions. The first assumption is that all travelers have perfect information. The second one is that the networks' demand is fixed, the users path-choices are fixed and determined before entering the network. These two assumptions may not be adequate to describe real-life situations. Therefore, the UE model was extended through three types of models and their combinations: stochastic (probabilistic) models, mixed strategy models and dynamic models. The stochastic models and the mixed strategy models relax the first assumption, and the dynamic models relax the second assumption.

Stochastic User-Equilibrium (SUE) Models

Stochastic user-equilibrium (SUE) models are well studied in the literature (Daganzo and Sheffi, 1977). The core of these models is the discrete route-choice models, according to which stochasticity results from variability between users' cost perception of networks. Mathematically, each user's perception is represented by an error term, which is a random variable that is distributed over the population of users. SUE models assume that the mean of the error term is zero, implying that on average, there is no difference between users. Moreover, the predicted network flows in SUE models are sensitive to the assumed distribution of the error term.

Mixed Strategy Path-Choice Models

A different probabilistic approach that also aims to describe users' variability is the mixed-strategies framework, was proposed by Koutsoupias and Papadimitriou (1999).The mixed strategy path-choice models approach stems from both the mixed-strategy approach of classic game theory and contemporary approaches of algorithmic game theory. According to this framework, users have (nearly) perfect information about the network links' travel times. This approach motivates them to choose a distribution over his or her pure path-choices strategies, as explained next.

Mixed Strategy Path-Choice Motivation

The main criticism against mixed strategies is that idea that users determine their actions, or in our case their traveling paths, on the outcome of a lottery, is against our intuition [Aumann, 1987] [Radner et al., 1982] [Rubinstein, 1991]. A rational analyst will be reluctant to believe that traveling decisions are made at random, and will prefer any other explanation that may point a reason for the users' choices.

However, according to Rubinstein (1991), mixed strategies, similarly to the error term in discrete choice models, can be interpreted as a "plan of action" which depends on the user's individual considerations that are not specified in the model. As a result, the users' behaviour seems to be random, although it is actually deterministic, similarly to the users' under SUE assumptions.

According to Rubinstein's interpretation to mixed strategies, a given user is not “flipping a coin” before each time he leaves his house to determine his traveling path. Instead, he is choosing deterministically a traveling path according to his own considerations, preferences, and biases. These decisions over a period of time may be seemed random, since each time a different choice can be made according to the user's private information at that given time. Observing these decisions over time enables us to measure their frequency, or in other words to assign each path-choice a probability. These probabilities are in fact the user's mixed strategies.

Rubinstein (1991) offers another interpretation to mixed strategies in non-cooperative recurring games for using them to explain variability among players. According to this interpretation, each player, or in our case - each network user, represent a large population of users. The users in each occurrence of the game are drawn randomly from these populations. Therefore, mixed strategies can be viewed as a distribution of pure (deterministic) choices in the population. Thus, reflecting inherit variability among the users' population.

Motivated by these interpretations, the foundation of any mixed strategy path-choice model is presented - the users' travel time function (the equivalent to "disutility" function in the game theory field).

Estimated Travel Time Function

Koutsoupias and Papadimitriou (1999) analyzed a simple case of single O-D network with parallel links, of which the congestion-dependent travelling times are linear (also known as KP model). The KP model is based on choosing the strategy that minimized the theoretical expected travel time of each link *e*, $E[l_e(f_e)]$. This model, which was based on two parallel links was later extended and studied mainly with respect to problems that rise in the field of computer sciences and electrical engineering (see [Czumaj et al, 2002]). Therefore, most of these papers are focused on algorithmic aspects of the problem (i.e., the complexity of computing Nash equilibria and computing bounds on worst-case Nash equilibrium) [Gofer, 2008].

Another approach was proposed by Orda et al. (1993) for their splitting models. They calculate the links' travel times as an estimation of the expected flow on each link e, $l_e[E(f_e)]$. Note that when the latency function is not linear, $E[l_e(f_e)] \neq l_e[E(f_e)]$.

Since the $E[l_e(f_e)]$ expression proposed within the KP model is hard to compute in large scale networks, Gofer (2008) inferred that the average user in the network doesn't choose his path according to it. Therefore, based on Orda et al. (1993) estimation of the expected flow, the mixed strategy approach to the static traffic assignment problem was implemented. Within this framework, it was shown that using mixed strategy is the user best decision-making mechanism facing of the effect of other users' path-choices; i.e., choosing random travel paths distribution that minimizes his or her expected travel time is beneficial to the network user. Thus, instead of basing the path-choices on a criterion of perceived minimal path cost or most appealing path (as in the stochastic approach), in the mixed-strategy models, each network user determines his or her mixed strategy according to the principle of expected minimal travel time.

The Mixed Strategy User Equilibrium

Similarly to the UE and SUE approaches, it is assumed that because of the non-cooperative behavior of the users and owing to congestion effects, network flows will converge to an operating point, at which no user will benefit from changing his mixed strategy unilaterally. For static networks, this is the mixed strategy user equilibrium (MSUE). Note that the MSUE is not equivalent to the UE (Rosenthal, 1973). The relation between these two equilibria is based on the fact that Nash equilibrium converges to Wardrop equilibrium when the number of the identical users sharing the same utility functions becomes large (Haurie and Marcotte, 1985).

Objectives

We believe that the new approach of combining game theory principle (such as Nash equilibrium) into traffic assignment problem by using mixed strategies path choice can be beneficial for transportation sciences and transportation engineering. This paper will discuss the application of mixed strategies in this context. Since mixed strategies models proposed so far were developed only under static conditions, this paper introduces new model in which network users employ mixed strategies under dynamic conditions.
This paper suggests a new approach for predicting traffic patterns: the Dynamic Mixed Strategy User Equilibrium (DMSUE). This model is based on realistic assumptions about the users: acting selfishly, aware of the network topology and links performance functions, but do not have a perfect knowledge of other users' path-choices. Each user has only an estimation of the number of people travelling in the network. This estimation leads the user to choose his path using mixed strategy in order to maximize his or her utility (i.e., minimizes his or her expected travel time).
This paper extends previous works which assumed static conditions. The suggested model uses a dynamic point of view, in order to investigate the changes of the flow distribution, in the sense of the users' path-choices pattern, over time.
The rest of the paper consists of five parts. The first part presents preliminary assumptions. The second part defines and formulates the dynamic mixed strategy model. The third part formulates the Dynamic Mixed Strategy User Equilibrium (DMSUE) model and proves the existence of a solution. The fourth path discusses the properties of DMSUE path-choices and presents two illustrative examples. The last part of the paper presents a summary and discussion of the results.

PRELIMINARY ASSUMPTIONS

In this section, the foundations of non-cooperative transportation networks' assumptions and notations are laid. First, a discussion on network structure representation is presented. Note that the DMSUE assumptions are divided into four types, based on the network aspects they correspond to:

- Network structure's assumption - refers to the differences between paths in the network;

- Network flows' assumption - refers to the time in which the users enter the network;
- Network flow-dependence latency assumptions - refers to the properties of the networks' link performance functions;
- Mixed strategy path-choice mechanism assumptions - refer to the time in which the network users made their decisions, and the information these choices are based on.

These assumptions are presented in next sections, discussing each of the aspects above. Next, network flows and flow-dependent travel times are discussed.

Network Structure

Consider a directed network $G = (V, E)$, where V is the set of nodes, and E is the set of links (edges). Denote the set of origin nodes by $S \subseteq V$; the set of destination nodes by $D \subseteq V$; and the set of origin-destination (O-D) pairs by $OD = \{\{s, d\}: s \in S, d \in D\}$. Each (O-D) pair $\{s, d\}$ is connected is by a set of paths, denoted by P^{sd}. The collection of all paths in the path-set is defined as $P = \bigcup_{sd} P^{sd}$. We assume that the network's paths differ from one another in at least one link. Also, the users' paths assumed to be without loops.

Network Flows

Within the dynamic framework, we divide the continuous time into times intervals, of one time unit each. Every time interval is denoted by Δt_τ and defined as $\Delta t_\tau = (t_\tau, t_{\tau+1})$, where $\tau \in [0, T]$. We assume that in the beginning of each time interval τ, there are N_τ users who enter the network. We divide the users who enter the network also by their sources and destinations. For each user $i \in N_\tau$, his source and destination are denoted as s_i and d_i respectively.
In addition, we denote the flow on path p at time interval τ by $f_{p,\tau}$. Thus, the flow on link e on time interval τ is defined as:

$$f_{e,\tau} = \sum_{p \in P} f_{e,\tau} \delta_{e \in p} \qquad (1)$$

Where $\delta_{e \in p}$ is an indicator variable that equals 1 if link e belongs to path p, otherwise, it equals 0.

Link Performance Functions

The travel time on link $e \in E$ is defined as a function l_e : R→ R and denoted by Λ_e. It is dependent on the flow (congestion) on e, such that $\Lambda_e = l_e(f_e)$. Similarly, Λ_p denotes travel time on path P, and the flow-dependent travel time function of path $p \in P$ is denoted by $l_P(f_p)$. Λ_p is the sum of travel times on all the links that comprise this path:

$$\Lambda_{p,sd} = \sum_{e \in E} l_e(f_e) \delta_{e,p}^{sd} \quad \forall p \in P, \forall (s,d) \in OD \qquad (2)$$

Numerous studies [Davidson, 1966], [Fare et al.,1982], [Smeed, 1986], [Wardrop, 1968], [USDOT,1986], and [Wilson, 1991] have shown that the relationship between traffic speed and traffic volume possess the following mathematical characteristics.
Assumption 1 The travel time function $l_e(f_e)$ is assumed to be: Non-negative; Continuous; Strictly increasing with respect to the link flow; Convex; The travel time on a link is only a function on the flow on that link.

DYNAMIC MIXED STRATEGY PATH-CHOICE MODEL

In this section the dynamic mixed strategy path-choice model is formulated. First, mixed path-choices and network users' utilities (their expected travel times) are defined. Next, the estimated travel time is presented, and finally the best choice is discussed. Based on these formulations, the mixed strategy user (Nash) equilibrium is defined in the subsequent section.

Mixed Strategy Path-Choices

The DUE and DSUE approaches assume that network users choose one path from their origin to their destination and use it. This may hold under the assumption of perfect information in the DUE and under the assumptions of maximum utility in the DSUE model.
However, when facing with incomplete information regarding the choices of other network users, as explained in the previous sections, it is reasonable to assume, that users choose their paths at random with some probability. The path-choice of the users is manifested in the choice of this probability, which is referred to as mixed strategy. This way, instead of basing their path-choices on a criterion of perceived minimal path cost or most appealing path, it is assumed that each network user determines his mixed strategy according to the principle of estimated minimal travel time. This way, each user's path- choice distribution minimizes his estimated travel time.

Network Users Mixed Strategy Path-Choice Mechanism Assumptions

As mentioned before, there are two main assumptions of the users mixed strategy path-choice mechanism, refer to the time in which the network users made their decisions, and to the information these choices are based on.
Assumption 2 *Under the DMSUE framework, the users are assumed to choose their path-choices when they enter the network, and do not change them during their travel.*
The dynamic aspect of the DMSUE is related to the variation of the users' path-choices over time, which happens due to the change in the demand.
Assumption 3 *According to the DMSUE, a user estimates his/her travel time, based on the information of the other users' path-choices, both from the previous time interval and from the current one, regardless the path-choice of users who enter the network at future time intervals.*
The above assumption is accurate for single-source networks, but not correct for multiple O-D pairs. However, assuming that in congestion hours there are many users, this means that additional users entering the network will choose a different path, because of the spread of the users on the network and consequently updated travel times. In the dynamic model literature, this is an approximation of the real-life scenario is (see [Han, 2007], [Han, 2006], [Tatomir et al., 2004], [Tatomir et al., 2006]).
Nevertheless, in order to overcome this assumption, historical traffic data may be used to obtain a measure of the future congestion, as suggested in [Tatomir et al., 2009], [Ando et al.,2005] and [Weyns et al., 2007]. This approach is discussed later in this paper. The users' mixed strategy path-choices under the DMSUE framework is presented next.

Definition 1 A probability distribution $\Psi_{\tau,i}^{s_i d_i,p} = \left(\psi_{\tau,i}^{s_i d_i,1}, \ldots, \psi_{\tau,i}^{s_i d_i,|P^{s_i d_i}|}, \ldots, \psi_{\tau,i}^{s_i d_i,|P|}\right)$is called a mixed strategy of user $i \in N_\tau$ who enter the network at t_τ from source s_i to destination d_i, where $P^{s_i d_i}$ denotes the set of all loopless paths from source s_i to destination d_i. $\psi_{\tau,i}^{s_i d_i,p}$ denotes the probability that traveler i who enter the network at t_τ from s_i to d_i, uses path $p \in P$. Note that this paper considers only loopless paths, that is, paths that do not travel along a link or a node more than once.

The probability that user i chooses a path which does not connect his OD pair $\{s_i, d_i\}$ is 0, i.e. $\psi_{\tau,i}^{s_i d_i,p} = 0,\ \forall p \notin P^{s_i d_i}$. Moreover, $\psi_{\tau,i}^{s_i d_i,p} \in [0,1],\ \forall p \in P$, and $\sum_{p \in P} \psi_{\tau,i}^{s_i d_i,p} = 1,\ \forall i \in [1, N_t], \forall \tau \in [0, T]$.

In the same way, a mixed strategies of all other users entering the network from source s to destination d at t_τ, (i.e, $j \in [1, N_\tau]$, such that $j \neq i$) is denoted as $\Psi_{\tau,-i}^{sd}$.

Definition 2 The probability that random user who enter the network at t_τ from source s to destination d choose a certain path p (denoted as $\psi_\tau^{sd,p}$), is defined as follows:

$$\psi_\tau^{sd,p} = \begin{cases} \frac{\sum_{i=1}^{N_\tau} \psi_{\tau,i}^{sd,p}}{N_\tau}, if\ p \in P^{sd} \\ 0, \qquad otherwise \end{cases} \tag{3}$$

According to the last definition, the users' probability of choosing a certain link e is expressed as follows.

Definition 3 The probability that travelers who enter the network at t_τ from source s to destination d to use a certain link e (denoted as $\psi_\tau^{sd,e}$), can be expressed by:

$$\psi_\tau^{sd,e} = \sum_{p \in P^{sd}} \delta_{e \in p}\, \psi_\tau^{sd,p} \tag{4}$$

Definition 4 A probability distribution $\Psi_\tau^{sd} = \left(\psi_\tau^{sd,1}, \ldots, \psi_\tau^{sd,p}, \ldots, \psi_\tau^{sd,|P^{sd}|}\right)$ is called a mixed strategy profile of a user who travels from source s to destination d at t_τ .

A mixed strategy of all users who traveling the network at $t_\tau$from other sources or heading towards other destinations is denoted as Ψ_τ^{-sd}.

Definition 5 Assume that the set of all origin-destination pairs is denoted as follows: O-D= $\{(s^1, d^1), \ldots, (s^k, d^k)\}$. Then, the vector of mixed strategy path- choices of all users who enter at τ from every source to every destination in the network is defined as: $\Psi_\tau = \left(\psi_\tau^{s^1 d^1}, \ldots, \psi_\tau^{s^k d^k}\right)$. The vector Ψ_τ is also called a mixed strategy profile of the users entering at t_τ .

We use Q_τ to denote the set of all possible mixed strategy profiles for users who enter at t_τ , from every origin to every destination$\{s, d\} \in OD$.

We use $Q = Q_1 \times \ldots \times Q_\tau \times \ldots \times Q_T$ to denote the space of all the possible mixed strategies profiles, for all the time intervals.

Moreover, the users' travel times depend on the inflow in each link. Based on Orda (1993), we assume that each user who enters the network at t_τ can estimate the expected inflow at t_τ on each link, as defined next.

$$E_{f_e}(\Psi_\tau) := \sum_{i=1}^{N_\tau} \sum_{p \in P^{s_i d_i}} \delta_{e \in p}\, \psi_{\tau,i}^{s_i d_i,p} \tag{5}$$

Contrary to evaluating the expected link flow, estimating the expected travel time $E[l_e(f_e)|\psi]$ is much harder and it is also not intuitive to the average user, who has limited understanding of the theory of probability. As mentioned before, we assume that users will use a different measure for the links' travel times, that is, $l_e[E(f_e)|\psi]$, which is easier to compute and more intuitive.

Furthermore, in the dynamic case, the network flow includes both the number of users enter the network at t_τ and the occupancy of the network. Therefore, the path-choices made by users who entering the network at t_τ are based on the path-choices of the users that enter at previous intervals. Therefore, in order to formulate the DMSUE model, we define the "history" of each link as the number of users who entered link *e* and have not exited the link at the τ^{th} time interval (denoted by H_τ^e). In other words, the "history" is the sum over the number of users who enter the network at each time interval $\tau' \leq \tau$, choose the link *e*, and still use it. Mathematically, H_τ^e is defined as follows:

$$H_\tau^e = \sum_{\tau'=0}^{\tau-1} E_{f_e}(\Psi_{\tau'})\, \delta_{\tau'+T(\Psi_{\tau'})} \qquad (6)$$

Where T (Ψ_τ) denotes the average travel time experienced by the users entering at time τ. Thus, $\delta_{\tau'+T(\Psi_{\tau'})}$ equals to 1 if the users who enter the network at time interval τ' still use link *e* at the τ^{th} time interval, and 0 otherwise.

Note that different users at the same time interval may choose different path- choices. Consequently, their expected travel time may be also different. Therefore, the users' average over their expected travel times is computed, for each arrival time $t_{\tau'} \in [0, T]$ (and denoted by $T_{\tau'}$). Since $N_{\tau'}$ users enter the network at $t_{\tau'}$, the average of the expected travel time of users who enter the network at this time is expressed as follows:

$$T_{\tau'} = \frac{\sum_{e\in E} E_{f_e}(\Psi_{\tau'})\, l_e\left(E_{f_e}(\Psi_{\tau'}) + H_{\tau'-1}^e\right)}{N_{\tau'}} = \frac{\sum_{e\in E} E_{f_e}(\Psi_{\tau'})\, l_e\left(E_{f_e}(\Psi_{\tau'}) + \sum_{\tau''=0}^{\tau'-2} E_{f_e}(\Psi_{\tau''})\, \delta_{\tau''+T(\Psi_{\tau''})}\right)}{N_{\tau'}} \qquad (7)$$

Where $\tau'' \leq \tau' \in [0, T]$.

Equations (6) and (7) yield that the path-choices making is a recursive process. Furthermore, the "history" of the users who enter the network at t_τ depends only on the path-choices of the users who enter the network before t_τ (i.e., only on the path-choices $\Psi_{\tau'}, \forall \tau' \leq \tau$).

The "history" factor presented in expression (6) can be extended by taking future congestion under consideration based on traffic historical data, divided into the network's links and time intervals. In this paper we hereby consider the "history" only when path-choices are made before the current travel time, as in (6) and (7).

The Estimated Travel Time Function

So far, we formulated the occupancy of the network and the users' path-choice profiles. Having this formulation, we define the utility function, which under the TA problem is equivalent to the travel-time function. As explained above, we refer to an approximation of the travel-time function, based on Orda [Orda et al.,1993], for static assignment problem. We assume that each user can estimate the expected flow on each link, therefore, he (or she) can also estimate his (or her) travel time, by using the function defined as follows.

Definition 6 The estimated travel-time function of user $i \in [1, N_\tau]$ who enter the network at the τ^{th} interval is defined as $U_{\tau,i} : Q_\tau \rightarrow R$ such that:

$$U_{\tau,i}(\Psi_\tau) = U_{\tau,i}\left(\Psi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_{\tau}^{-s_i d_i}\right) =$$
$$\sum_{p\in P} \psi_{\tau,i}^{s_i d_i,p} \sum_{e\in p} l_e\left(1 + H_\tau^e + \sum_{p'\in P} \delta_{e\in p'} \sum_{j\neq i, j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'}\right) \qquad (8)$$

Where H_τ^e and N_τ are defined as in expression (6). Note that the definition above considers time-dependent flows, which are more general compared to Orda. Moreover, one can see that the utility is a sum over all possible paths, for each path, a sum over all the links belong to it are considered. For each link, we consider its latency function depends on the users' path-choices, both from the previous intervals and from the current interval.

Note that by adding the "History" expression H_τ^e to the estimated travel time, the FIFO condition takes place in the DMSUE. In this way, for each link, the estimated travel time of users entering the network at τ is longer than the "history", which is the estimated travel time left for the users who enters the network at previous time intervals to stay in the link.

Proposition 1 Given path-choices mixed strategy Ψ_τ , the estimated travel time function $U_{\tau,i}(\Psi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}), \forall i \in [1, N_\tau], \forall \tau \in [0, T]$ is a convex function with respect to Ψ_τ .

Proof $\tau \in [0, T], \forall i \in [1, N_\tau]$, the utility function $U_{\tau,i}(\Psi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i})$ is a convex combination of convex functions *$l_e(.)$*, according to the assumptions above. Therefore, all the utility functions are also convex.

3.3 "User's Best Response"

Definition 7 Given $\Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}$, the mixed strategy profile $\Psi_{\tau,i}^{sd*}$ is referred to as a best response of user *i* at time interval $\tau \in [0, T]$, if:

$$\Psi_{\tau,i}^{s_i d_i *} = \underset{\Phi_{\tau,i}^{s_i d_i} \in \mathbf{Q}_\tau}{\arg\min} \{U_{\tau,i}(\Phi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i})\} \qquad (9)$$

Where Q_τ denotes the space of all the possible mixed strategies profiles for users who enter the network at t_τ .

For each time interval $\tau \in [0, T]$, let $b_{\tau,i}^{s_i d_i}(\Psi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}) = \Psi_{\tau,i}^{s_i d_i *}$ be a mapping that finds, for user $i \in$ N, the set of best responses for a profile of mixed strategies $\Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}$ that is employed by the other users. A formal definition of this set of functions is presented below.

Definition 8 A best response mapping of user $i \in [1, N_\tau]$, at time interval $\tau \in [0, T]$ is mapping $b_{\tau,i}^{s_i d_i}; \prod_{j \in N_\tau} Q_\tau \to [0,1]^{|P|}$ defined as follows:

$$b_{\tau,i}^{s_i d_i}(\Psi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}) = \Psi_{\tau,i}^{s_i d_i *} \qquad (10)$$

Note that each one of the N_τ users arriving at the same time interval $\tau \in [0, T]$, is a non-cooperative user, who chooses one of his best responses, based on the others' path-choices. For each time interval, the process of choosing a best path-choice by every user can be considered as a stand- alone stage modelled as a static case.

Therefore, the DMSUE model examines multi stage process of static cases (based on the MSUE model). At each stage, the static case modulation takes under account the results (path-choices) of the previous stages.

Corollary 1 For each time interval $\tau \in [0, T]$, the set of the best response mappings $\Psi_{\tau,i}^{s_i d_i *}$ matches to the best response mapping $\{b_i(\Psi_i)\}$ in the MSUE (the static case) while the network occupancy is based on the path-choices that were made at the previous time intervals.

The properties of the best response mapping are described next. We consider these properties in the followed discussion, referring to the Dynamic Mixed Strategy User Equilibrium (DMSUE). From the last proposition we may obtain the following.

Corollary 2 In a transportation network that satisfies our assumptions, for each time interval $\tau \in [0, T]$, given a strategy profile of the users $\Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}$,user $i \in [1, N_\tau]$ always has a best response.

THE DMSUE PROBLEM

This section consists of two parts. First, the DMSUE is defined and the conditions of its existence are discussed, and the existence of at least one specific solution is proved. Next, the existence of a specific DMSUE in general networks is analyzed, based on KKT method.

DMSUE Definition

As describe above, the DMSUE model examines multi stage process of static cases. Thus, the Dynamic Mixed Strategy (Nash) User Equilibrium (DMSUE) defined as a multi stage process of Instantaneous Mixed Strategy (Nash) User Equilibrium (IMSUE), which is described as follows.

Definition 9 Given a time interval $\tau \in$ [0, T], an Instantaneous Mixed Strategy (Nash) User Equilibrium (IMSUE) is the mixed strategy profile $\Psi* = (\Psi_{\tau,1}^{s_1 d_1 *}, \dots, \Psi_{\tau, n_\tau^{s_1 d_1}}^{s_1 d_1 *}, \dots, \Psi_{\tau, n_\tau^{s_k d_k}}^{s_k d_k *})$ in which $\forall$i$\in$ [1, Nτ]:

$$\Psi_{\tau,i}^{s_i d_i *} = \left\{ \underset{\Phi_{\tau,i}^{s_i d_i} \in \mathbf{Q}_\tau}{\arg\min} \{ U_{\tau,i}(\Phi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}) \} \;\middle\|\; \begin{array}{l} \sum_{p \in \mathbb{P}} \psi_{\tau,i}^{s_i d_i, p} = 1 \\ \psi_{\tau,i}^{s_i d_i, p} \in [0,1], \forall p \in P^{s_i d_i} \\ \psi_{\tau,i}^{s_i d_i, p} = 0, \forall p \notin P^{s_i d_i} \end{array} \right\} \quad (11)$$

Where κ =|OD|, and $n_\tau^{s_1 d_1}$ denotes the number of users enter the network at t_τ from origin s_i to destination d_i.

The instantaneous mixed strategy equilibrium (IMSUE) definition can be associated to the best response mapping, as shown next.

Corollary 3 Given a time interval $\tau \in$ [0, T], an instantaneous mixed strategy (Nash) equilibrium profile is achieved, when each one of the users choose his or her best response (i.e., for every user i with OD pair $\{s_i, d_i\}$, who enter the network at t_τ , $\Psi_{\tau,i}^{s_i d_i *}$ is the solution of the optimization problem presented in τ (11)).

Note that in the optimization problem presented in (11)), there are continuously differentiable convex objective function $U_{\tau,i}(\Psi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i})$ and inequality constraints, with linear equality constraints. Thus, it has at least one solution (that is one way to prove corollary 2).

However, the last corollary does not prove the convergence of all best responses into an equilibrium point. This is proved in the next theorem.

Theorem 1 A network congestion game (or transportation network) that follows our assumptions (i.e, a game in which the users who enter the network in each time interval $\tau \in$ [0, T] choose their paths, based on their estimated travel-time function (definition 6)), admits at least one instantaneous mixed strategy user equilibrium (IMSUE).

Proof The existence of the IMSUE cannot be proved by Nash's theorem [Nash, 1951], since in our model the users do not choose their path-choices based on the expected travel time, but based on their *estimated travel time*, as in equation (8). A network that satisfies our assumptions (N_τ users who make their path-choices based on the estimate travel time function) has Nash equilibrium. In addition, each time interval $\tau \in$ [0, T], is a special case of the static transportation network, in which the network occupancy is based on the previous users path-choices. Therefore, we conclude that in a network that satisfies our assumptions; at least one IMSUE operating point exists.

So far we defined instantaneous mixed strategy equilibrium, based on the static case. An extension is needed in order to include dynamic into it, as defined next.

Definition 10 A mixed strategy profile of all the time intervals in a checked period $\tau \in [0, T]$, $\Psi^* = (\Psi_1^*, ..., \Psi_\tau^*, ..., \Psi_{|T|}^*)$ is a Dynamic Mixed Strategy (Nash) User Equilibrium (DMSUE) if $\forall \tau \in [0, T]$, Ψ_τ^* is an Instantaneous τ Mixed Strategy (Nash) Equilibrium. Mathematically, a Dynamic Mixed Strategy (Nash) Equilibrium (DMSUE) is defined as follows.

$$\Psi_{\tau,i}^{s_i d_i *} = \left\{ \arg\min_{\Phi_{\tau,i}^{s_i d_i} \in \mathbf{Q}_\tau} \{U_{\tau,i}(\Phi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i})\} \;\middle\|\; \begin{array}{l} \sum_{p \in \mathbb{P}} \psi_{\tau,i}^{s_i d_i, p} = 1 \\ \psi_{\tau,i}^{s_i d_i, p} \in [0,1], \forall p \in P^{s_i d_i} \\ \psi_{\tau,i}^{s_i d_i, p} = 0, \forall p \notin P^{s_i d_i} \end{array} \right\} \quad (12)$$

Similarly to the instantaneous mixed strategy (Nash) equilibrium, there is equivalent definition of the dynamic mixed strategy (Nash) equilibrium in terms of best responds.

Corollary 4 DMSUE is a path-choices profile in which for every time interval $\tau \in [0, T]$, each one of the users $i \in [1, N\tau]$ choose his best respond.

Note that the choices (the best responses) of users who enter at t_τ depend on the occupancy of the network. This occupancy depends on the path-choices made by the users who enter at the previous time intervals $\tau' \leq \tau$. Therefore, the best responds of the users who enter the network at t_τ is based on the best responds of the users who enter it at $t_{\tau'}$ (where $\tau' < \tau$).

Theorem 2 A network congestion game that follows our assumptions, admits at least one Dynamic Mixed Strategy User Equilibrium (DMSUE).

Proof Under the assumptions that the users who enter the network in each time interval $\tau \in [0, T]$ choose their paths, based on their estimated travel- time function, we proved that there exists an IMSUE point (theorem 1). This equilibrium of the users who enter at each time interval τ does not vary with time, (since we assume that the users' path-choices are made only before they enter the network). In this way, all the users who enter the network at each time interval create a mixed strategy equilibrium profile (at least one profile). Thus, at least one DMSUE exists.

Note that the DMSUE is a nonlinear convex optimization problem (also referred as "convex program"), since it's a multi-stage IMSUE problems (which are all convex). Moreover, this problem can be written as follows:

$$\begin{aligned} & \textit{minimize} \quad U_{\tau,i}(\Psi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}) \\ & \textit{subject to} \\ & \qquad \sum_{p \in \mathbb{P}} \psi_{\tau,i}^{s_i d_i, p} = 1, \\ & \qquad \psi_{\tau,i}^{s_i d_i, p} \in [0,1], \ \forall p \in P^{s_i d_i} \\ & \qquad \psi_{\tau,i}^{s_i d_i, p} = 0, \forall p \notin P^{s_i d_i} \end{aligned} \qquad (13)$$

The optimization problem can be rewritten as:

$$\begin{aligned} & \textit{minimize} \quad U_{\tau,i}(\Psi_{\tau,i}^{s_i d_i}, \Psi_{\tau,-i}^{s_i d_i}, \Psi_\tau^{-s_i d_i}) \\ & \textit{subject to} \\ & \qquad \sum_{p \in \mathbb{P}} \psi_{\tau,i}^{s_i d_i, p} = 1 \\ & \qquad \sum_{p \notin P^{s_i d_i}} \psi_{\tau,i}^{s_i d_i, p} = 0 \\ & \qquad \psi_{\tau,i}^{s_i d_i, p} \geq 0 \quad \forall p \in \mathbb{P} \end{aligned} \qquad (14)$$

Since the optimization problem above (14) is convex, we can use the KKT method in order to find its solution, as shown in the next section.

DMSUE Analysis based on KKT Method

Within the KKT method, an optimization problem is divided into two different parts. The first part is the objective function and second one includes the constraints. In optimization problem (14), the objective function is as follows:

$$U_{\tau,i}(\Psi_{\tau,i}^{s_id_i}, \Psi_{\tau,-i}^{s_id_i}, \Psi_{\tau}^{-s_id_i}) = \sum_{p\in\mathbb{P}} \psi_{\tau,i}^{s_id_i,p} \cdot \sum_{e\in p} l_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p'} \sum_{j\neq i,j=1}^{N\tau} \psi_{\tau,j}^{s_jd_j,p'}) \qquad (15)$$

Note that the above expression is nonlinear since the link performance functions $l_e(\cdot)$ are nonlinear functions. The constraints are divided into three sets. The first set includes part of the equality constraints in (14) optimization problem, denoted as a set of functions $\{h_{\tau,i}\left(\Psi_{\tau,i}^{s_id_i}\right)\}_{i\epsilon[1,N_\tau]}: [0,1] \to \mathbb{R}$, is defined as follows:

$$h_{\tau,i}(\Psi_{\tau,i}^{s_id_i}) := \sum_{p\in\mathbb{P}} \psi_{\tau,i}^{s_id_i,p} - 1 = 0 \qquad (16)$$

The second set equality constraints in the problem, is denoted as a set of functions $\{g_{\tau,i}\left(\Psi_{\tau,i}^{s_id_i}\right)\}_{i\epsilon[1,N_\tau]}: \mathbb{P} \to \mathbb{R}$, is defined as follows:

$$g_{\tau,i}^{p}(\Psi_{\tau,i}^{s_id_i}) := \sum_{p\notin P^{s_id_i}} \psi_{\tau,i}^{s_id_i,p} = 0 \qquad (17)$$

In addition, there are $N_\tau \cdot \mathbb{P}$ non-negative constraints:

$$\psi_{\tau,i}^{s_id_i,p} \geq 0 \qquad (18)$$

Note that the standard optimization problem presented in (14) sustains the KKT conditions; both the objective function and the constraints are continuously differentiable. The Lagrangian function is constructed by adding the constraints to the objective function as follows:

$$\mathbb{L}_{\tau,i}(\Psi_\tau, \lambda_{\tau,i}^{s_id_i}, \mu_{\tau,i}^{p'}) = \qquad (19)$$

$$= U_{\tau,i}(\Psi_{\tau,i}^{s_id_i}, \Psi_{\tau,-i}^{s_id_i}, \Psi_{\tau}^{-s_id_i}) + \lambda_{\tau,i} \cdot h_{\tau,i}^{s_id_i}(\Psi_{\tau,i}^{s_id_i}) + \mu_{\tau,i}^{p} \cdot g_{\tau,i}^{p}(\Psi_{\tau,i}^{s_id_i}) =$$

$$= \sum_{p\in\mathbb{P}} \psi_{\tau,i}^{s_id_i,p} \cdot \sum_{e\in p} l_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p'} \sum_{j\neq i,j=1}^{N\tau} \psi_{\tau,j}^{s_jd_j,p'}) +$$

$$+ \lambda_{\tau,i} \cdot (\sum_{p\in\mathbb{P}} \psi_{\tau,i}^{s_id_i,p} - 1) + \mu_{\tau,i}^{p} \cdot \sum_{p\notin P^{s_id_i}} (\psi_{\tau,i}^{s_id_i,p}) \qquad (20)$$

According to the KKT method, the optimal (minimal) value of the objective function is the solution of the following equation:

$$\nabla \mathbb{L}_{\tau,i}(\Psi_\tau, \lambda_{\tau,i}^{s_id_i}, \mu_{\tau,i}^{p}) = 0 \qquad (21)$$

Next, the lagrangian function is differentiated by each set of variables.

$$\frac{\partial \mathbb{L}_{\tau,i}}{\partial \lambda_{\tau,i}} = \sum_{p\in\mathbb{P}} \psi_{\tau,i}^{s_i d_i,p} - 1 = 0$$
$$\frac{\partial \mathbb{L}_{\tau,i}}{\partial \mu_{\tau,i}^{p}} = \sum_{p\notin P^{s_i d_i}} (\psi_{\tau,i}^{s_i d_i,p}) = 0$$
$$\frac{\partial \mathbb{L}_{\tau,i}}{\partial \hat{\psi}_{\tau,i}^{s_i d_i,p''}} |_{p''\in P^{s_i d_i}} = \sum_{e\in p''} l_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p'} \sum_{j\neq i,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'}) + \lambda_{\tau,i} = 0$$
$$\frac{\partial \mathbb{L}_{\tau,i}}{\partial \hat{\psi}_{\tau,i}^{s_i d_i,p''}} |_{p''\notin P^{s_i d_i}} = \sum_{e\in p''} l_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p'} \sum_{j\neq i,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'}) + \lambda_{\tau,i} + \mu_{\tau,i}^{p''} = 0$$
$$\frac{\partial \mathbb{L}_{\tau,i}}{\partial \hat{\psi}_{\tau,j}^{s_j d_j,p''}} |_{j\neq i} = \sum_{p\in\mathbb{P}} \psi_{\tau,i}^{s_i d_i,p} \sum_{e\in p} \delta_{e\in p''} l'_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p'} \sum_{j\neq i,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'}) = 0 \quad (22)$$

Based on the partial derivatives computation in (22), we conclude that the KKT multipliers $\mu_{\tau,i}^{p} = 0$, $\forall i \in [1, N_\tau]$. Furthermore, if all the KTT multipliers are zeros (i.e, $\lambda_{\tau,i}$, $\forall i \in [1, N_\tau]$, $\forall p' \in \mathbb{P}$), the trivial solution will achieved; Given two arbitrary users i, k, who enter at the same time interval τ and share the same OD pair, the analysis conducted in (22) suggested that:

$$\frac{\partial \mathbb{L}_{\tau,i}}{\partial \hat{\psi}_{\tau,i}^{s_i d_i,p''}} |_{p''\in P^{s_i d_i}} = \sum_{e\in p''} l_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p} \sum_{j\neq i,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'}) + \lambda_{\tau,i} = 0$$
$$\frac{\partial \mathbb{L}_{\tau,k}}{\partial \hat{\psi}_{\tau,k}^{s_k d_k,p''}} |_{p''\in P^{s_k d_k}} = \sum_{e\in p''} l_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p} \sum_{j\neq k,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'}) + \lambda_{\tau,k} = 0 \quad (23)$$

Since the users share the same OD pair $\{s_i, d_i\} = \{s_k, d_k\}$. For each $e \in p''$

$$[l_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p} \sum_{j\neq i,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'}) - l_e(1 + H_\tau^e + \sum_{p'\in\mathbb{P}} \delta_{e\in p} \sum_{j\neq k,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'})] + \\ + [\lambda_{\tau,i} - \lambda_{\tau,k}] = 0 \quad (24)$$

The combination of equations (23) and (24) with the constraints in (22) yields the trivial solution for the optimization problem, according to KKT. That is, $\lambda_{\tau,i}$ = 0, $\forall i \in [1, N_\tau]$, and therefore:

$$\sum_{j\neq i,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'} = \sum_{j\neq k,j=1}^{N_\tau} \psi_{\tau,j}^{s_j d_j,p'} \quad (25)$$

Thus, we proved that the trivial DMSUE solution which exists in every network is achieved where all the other users who enter the network at t_τ, with the same OD pair (presented as user j), share also the same path-choices, $\Psi_{\tau,i}^{s_i d_i, p'}$ $\forall p' \in \mathbb{P}$. This holds for every arbitrary users i , k with the same OD pair. Therefore, the trivial solution is that all the users who share the same OD pair and the entrance time to the network τ, have the same path-choices.

In particular, we proved that the optimization problem in (14) has at least one solution. As a result, there exists at least one IMSUE for every $\tau \in$ [0,T], hence, there exists at least one DMSUE.

The solution of the mathematical program represents the equilibrium point, in which all the users who enter the network at the same time and share the same source and destination - have the same path-choices. This has an intuitive interpretation: all the users enter at the same time with the same OD pair, share the same knowledge, as discussed next.

Symmetric Equilibrium

As mentioned before, we assume that $n_\tau^{s_i,d_i}$ users enter the network at the beginning of every time interval $\tau \in [0, T]$ from source s_i to destination d_i. It is reasonable to assume that each one of the $n_\tau^{s_i,d_i}$ users who enter the network τ at a given time and share O-D pair $(\{s_i, d_i\})$, have the same available path set P^{s_i,d_i} and the same estimated travel-time function. Therefore, they make their path-choices based on the same optimization problem presented in (12). Users who share the same estimated travel time (utility) functions are also referred to as *symmetric users.* A game with symmetric users is called *symmetric game.* Mathematically, symmetric game is defined as follows:

Definition 11 [Chang et al.,2004] A normal-form game is symmetric if all users have the identical strategy spaces and the same utility function: $u_i(\Psi_i, \Psi_{-i}) = u_j(\Psi_j, \Psi_{-j}), \forall i \neq j \in I$.

Corollary 5 According to the trivial DMSUE, given a directed network G = (E,V), all the $n_\tau^{s_i,d_i}$ users who enter the network each time interval and share the same properties (s_i, d_i, τ) (i.e., source, destination and arrival time, respectively) are considered as symmetric users, since: $U_{\tau,i}(\Psi_{\tau,i}^{s_i,d_i}, \Psi_{\tau,-i}^{s_i,d_i}, \Psi_\tau^{-s_i,d_i}) = U_{\tau,j}(\Psi_{\tau,j}^{s_j,d_j}, \Psi_{\tau,-j}^{s_j,d_j}, \Psi_\tau^{-s_j,d_j})$ where $i \neq j \in [1, n_\tau^{s_i,d_i}]$

Therefore, their path-choice profile is symmetric as well, and referred as a symmetric profile. A symmetric profile which satisfies Nash equilibrium conditions is called a symmetric equilibrium.

The DMSUE Uniqueness

Assume that every $n_\tau^{s_i,d_i}$ users enter the network at each time interval τ, are symmetric, the DMSUE exists. Next, under this assumption, its uniqueness is discussed.

Corollary 6 Under the assumptions of the IMSUE, a single OD network, with symmetric users at each time interval $\tau \in [0, T]$, has a unique IMSUE.

Proof According to Gofer [Gofer,2008], a static single OD network that satisfies our assumptions has a unique Nash equilibrium. In addition, each time interval $\tau \in [0, T]$, is special case of the static transportation network, in which the network occupancy is based on the previous users path-choices. Therefore, we conclude that a single OD network that satisfies our assumptions has a unique IMSUE operating point. Moreover, we may conclude that in single OD networks, there is a unique DMSUE.

Corollary 7 A single OD network congestion game that follows our assumptions has a unique DMSUE.

Proof Based on corollary 6, under our assumptions, a single OD network has a unique IMSUE. Therefore, at each time interval, the users who enter the single OD network create a unique mixed strategy equilibrium profile. Thus, the DMSUE is unique.

ILLUSTRATIVE EXAMPLES

In this section, two examples will be used to demonstrate and discuss the properties of DMSUE. The first example is the well-known Braess paradox network [Braess et al., 2005]. The second example is a simple grid network. In order to obtain the users' path-choices (the flow), a DMSUE simulation was developed. The simulation solves the optimization problem for each time interval τ and demand level N_τ, according to the optimization of the previous time interval (τ– 1). For each example, the change in the users' path-choices with respect to the demand level was examined.

Example 1 – Braess' Network

Consider a network with 4 nodes, 5 links, and a single OD pair (1– 4), which is connected by three paths. These paths are denoted by their nodes 1– 2– 4, 1– 3– 2– 4 and 1– 3– 4, and also as p_1, p_2, p_3 respectively (see figure 4). The network links' performance function follows the BPR link travel time, where the parameters of each link are presented in table 1.

Table 1 – Braess' Network's Link Parameters

Link	Connected nodes	Free-flow travel time	capacity
1	1-2	3	50
2	2-4	2	100
3	1-3	2	100
4	3-4	3	50
5	3-2	1	70

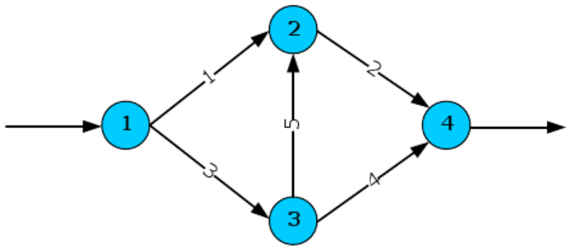

Figure 1 – Braess' Network

The users' path-choice observed in this example are presented in figure 2, for two different demand levels n = 55, n = 62. The X-axis represents time intervals, and the Y-axis represents the path-choice probability of choosing paths p1 and p2 by users who enter the network at each time interval. Note that in this case, for every time interval τ, for every user *i*, there are three paths the user can choose from (i.e, $\forall\tau\in [0, T], \forall i\in [1, N_\tau^{1-4}]$, $\psi_{\tau,i}^{p_1}+\psi_{\tau,i}^{p_2}+\psi_{\tau,i}^{p_3} = 1$). Therefore, it is sufficient to show only the probabilities for p_1 and p_2, as shown in figures 2.

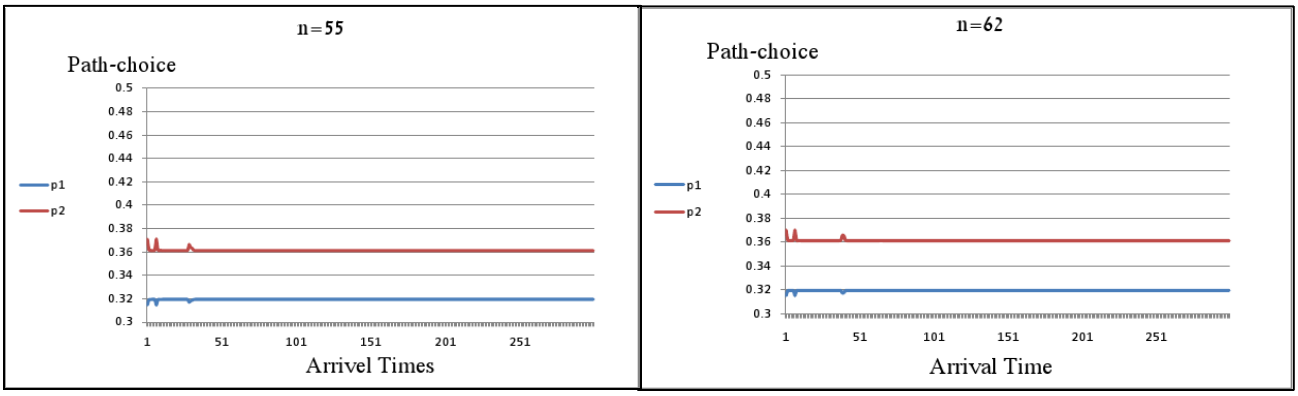

Figure 2 – Braess' Network – Convergence of Users' Path Choice over Time

Figure 2 shows that the user's paths-choices over time intervals converge to p_1=32%, p_2=36%, and consequently p_3=32%. This convergence matches the path proportions for the static case. The difference between the travel times achieved by the users, is presented in Figure 3 (for n = 55, in the left, and for n = 62, in the right).

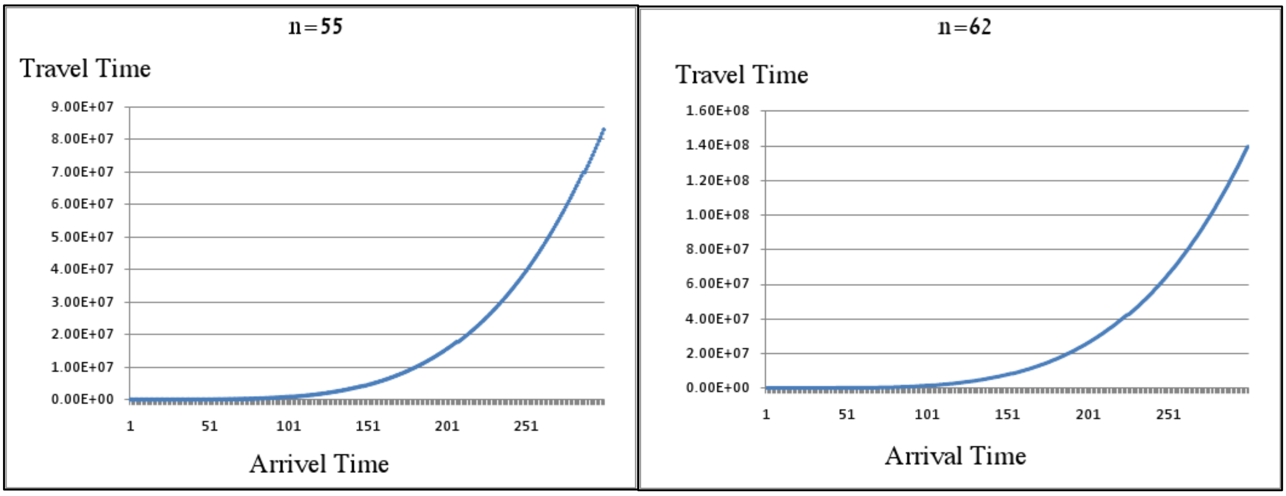

Figure 3 – Braess' Network's Travel Times

Example 2 – Grid Network

Consider a network with 6 nodes, 7 links, and a single OD pair - 1–6, which is connected by two paths. These paths are defined as $p_1 = 1-2-3-6$, $p_2 = 1-4-5-6$ and $p_3 = 1-2-5-6$. The network links' performance function follows the BPR link travel time, where the parameters of each link are presented in table 2.

Table 2 – Grid Network's Link Parameters of

Link	Connected nodes	Free-flow travel time	capacity
1	1-2	1	50
2	2-3	2	70
3	1-4	5	70
4	4-5	5	50
6	5-6	3	100
7	3-6	2	100

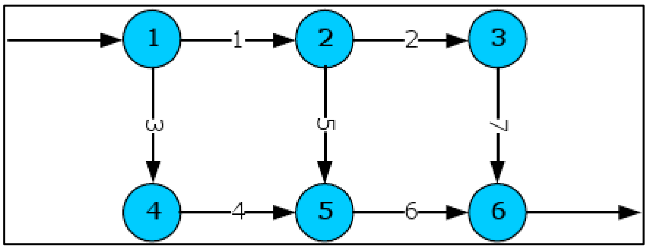

Figure 4 – Grid Network

The users' path-choices over time are presented in figure 5, for two different demand levels: n=50, and n=100.

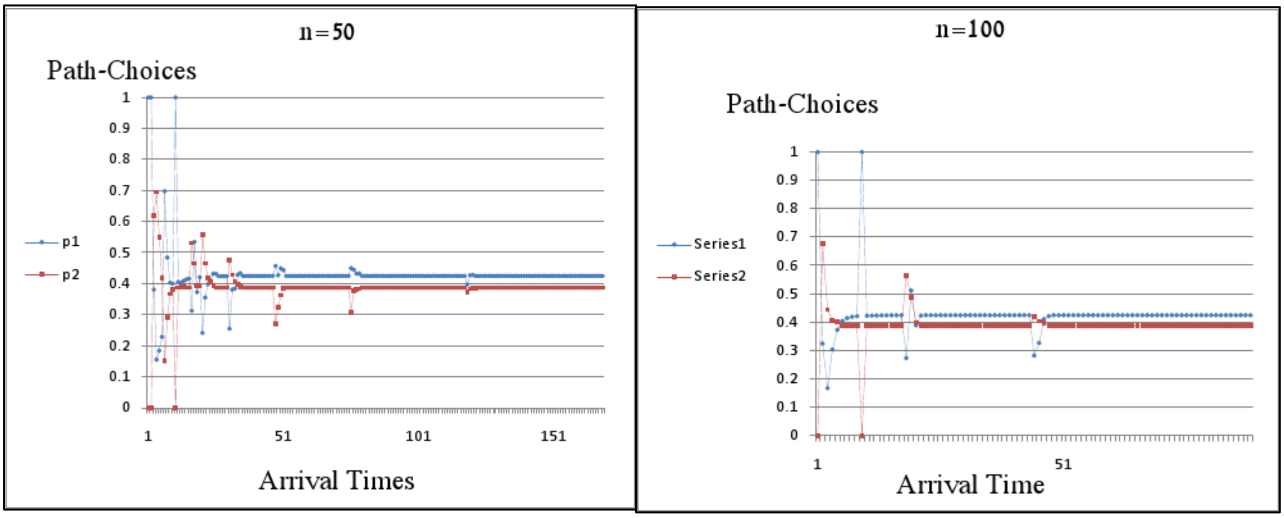

Figure 5 – Grid Network – Convergence of Users' Path Choices over Time

Similar to the Braess Network case, Figure 5 shows that the user's paths-choices over time intervals converge to p_1=42%, p_2=39%, and consequently p_3=19%. This convergence matches the path proportions for the static case. The difference between the travel times achieved by the users, is presented in figure 6 (for n = 50, in the left, and for n = 100, in the right).

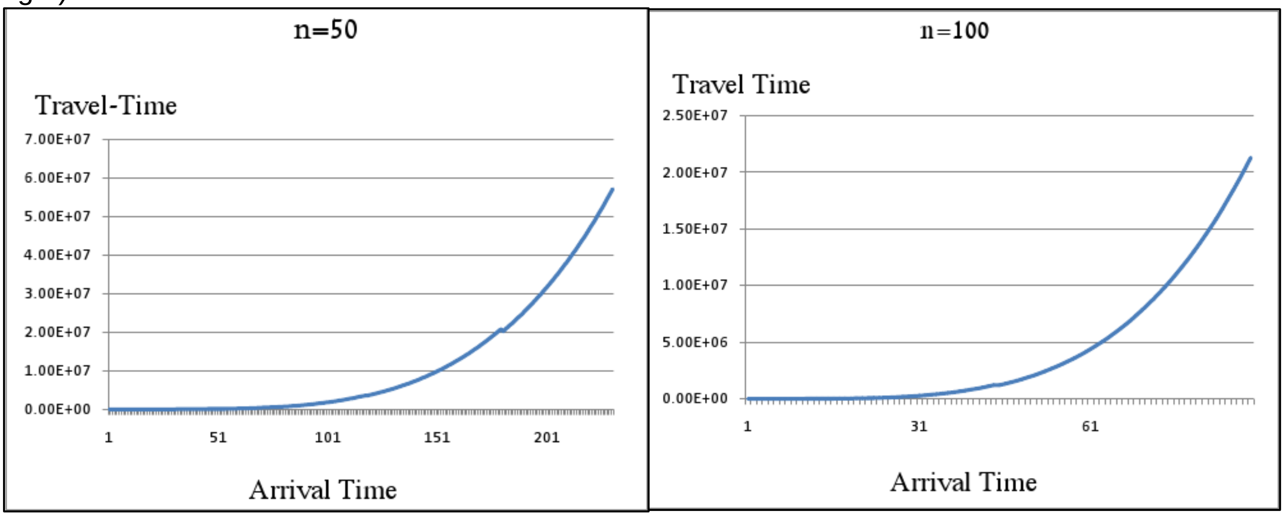

Figure 6 – Grid Network – Travel Times

SUMMARY AND DISCUSSION

In this paper, we developed the DMSUE framework for the solution of the dynamic traffic assignment problem. This approach assumes that users, because of contemporary transportation information systems, have better information about the variables that affect their path choices, such as travel times and costs. The main source of uncertainty that remains is related to the path choices of other users. We assume that the network's users cope with this uncertainty by using mixed strategies - they choose their paths at random according to some path-choice probability. A given user determines his path-choice probability according to a criterion of expected travel-time minimization, while taking into consideration the path-choice probabilities of both the users who enter the network at the same time and the users who enter the network before and still exist there. Moreover, we suggested a way to extend this assumption, by taking under consideration the future congestion based on traffic historical information.

Another assumption that was made within the DMSUE framework is that each user chooses his path-choice only when he enters the network. In order to overcome this assumption, another approach can be used, as multi-stage DMSUE, for each of the links. This way, before entering the network, the user would choose his path-choice, based on the current and the history data. Then, when he enters to each of the intersections, he would estimate his travel time again, and choose his path-choice, based on more updated information. This type of assignment also referred as Markov assignment (see Cascetta, E., 1989)). and Hazelton, 1998).

However, this new approach's calculation may have higher complexity, since each of the users who enter the network at the same time interval are now divided into all the possible links choose, in each intersection. Therefore, instead of analyzing only the number of paths for each time interval, according to the new approach, the number of possible links will take into consideration, for every intersection, for each time interval. The complexity is therefore increases in each time interval for each node in the network by the number of possible links emanating from the node. Furthermore, such approach must prevent users from travel in loops. Thus, also the set of links each of the users pass should also take under consideration while choosing a new path-choice in every intersection.

Under the DMSUE assumptions, an operating point of the system at a single time interval, at which no user has an incentive to choose a different mixed strategy - all path choice probabilities provide the "best response" to the other users' mixed strategies, is defined as instantaneous mixed strategy user equilibrium (IMSUE). This definition is extended to the dynamic mixed strategy user equilibrium (DMSUE) by consider the IMSUE operating point of each time interval.

An investigation of the mathematical properties of the DMSUE yields the following: (i) A network that follows our assumptions has a DMSUE operating point; (ii) A symmetric DMSUE exists, referring to each OD pair - i.e, all users traveling between any given pair of source-destination nodes, with the same entrance time, apply the same mixed strategies in DMSUE.

A comparison of network performance in DMSUE with the traditional DUE and dynamic SUE models yields some differences. The main difference lies in path travel times (or, more precisely, expected path travel times): in DMSUE, they do not follow Wardrop's first principle. That is, in DMSUE it is possible that the travel times of some of the paths used will be different and not equal. This results from the path-choice mechanism, which is based on users choosing their "best response" to other users' path-choices, rather than on equalizing the paths travel times as in the DUE and dynamic SUE approaches.

As mentioned before, Haurie and Marcotte (1985) established the fact that Nash equilibrium converges to Wardrop equilibrium when the number of the identical users sharing the same utility functions become large. Therefore, in each time interval, the IMSUE will converge to the UE. Thus, the DMSUE will converge to DUE models.

This paper lays the foundation for the DMSUE approach. It is also important to address other issues: applying this framework to different networks' topologies, formulating and analyzing developing the DMSUE flow path-choices patterns and the travel times distribution.

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