A NEW SEDIMENT MODEL OF TRAFFIC INTENSITY

Bauyrzhan YEDILBAYEV, University of Buckingham,UK, b.t.yedilbayev@gmail.com
Arnold BRENER, State University of South Kazakhstan, RK, amb_52@mail.ru
Orazaly BALABEKOV, State Pedagogical Institute, RK, oreke2004@mail.ru

ABSTRACT

The work deals with the mathematical modelling of traffic phenomena. The submitted model is based on prospective analogy of the some described phenomena with particles sedimentation. Both the qualitative analysis of the model and the numerical experiment are carried out. Results of research have been compared with the known data of supervision of the traffic on city highways. The offered model and appropriate equations can be used at designing of highways, reconstruction of roads and highways, for exact regulations of traffic lights, and also for working out actions for preventing the jams on roads which are one of principal causes of air pollutions and discomfort of inhabitants of cities.

Keywords: traffic flow, sediment model, traffic intensity, hysteresis, traffic jam.

1 INTRODUCTION

Improvement of the reliability of transportations as well as their security, quality with ensure of the environmental characteristics are extremely relevant issues of today. Solution of these problems requires an increase in the cost of infrastructural improvements of the transport network for turning it into a flexible and well controlled logistics system (Greenshields, B.D., 1934; Gasis, D.C. et al*.*, 1959; Cremer, M., Ludwig, J., 1986).
However, the investment risk is greatly increased if you do not take into account the laws of development of the transport network and balance of load on its sites. Ignoring these laws leads to the formation of traffic jams, to overloading or underloading the individual lines and nodes in the network, as well as increasing the level of accidents and environmental damage.
In order to identify effective strategies for traffic management in the city, the optimal solutions for designing the road network and traffic management it should be taken into account a wide range of characteristics of the traffic flow and the influence of external and internal factors on the dynamic characteristics of the traffic flows. There exist now various theories of traffic flows developed by researchers of different fields of knowledge - physicists, mathematicians and specialists in operations research (Greenberg, H., 1959; Haight, F.A., 1963; Cremer, M., Ludwig, J., 1986).

A research experience in this topic is great. However, the overall level of the reached results in trying to their practical application is often not sufficient. In our opinion this can be explained by the following factors.

Traffic flow is unstable and diverse in its nature, and the objective information about it is the most complex and resource-intensive part of the control system.

The quality criteria for traffic control are contradictory. It is conditioned by need to ensure the smooth flow while reducing the costs of movement by imposing restrictions on the speed and directions of travel.

Enforcement of traffic management in practical implementation is always inaccurate, and, as a result, the road leads to unintended effects.

Thus, the difficulty of formalizing the regularities of traffic flow has become a major cause of the backlog of research results from the requirements of practice.

The first macroscopic model, in which the traffic regularities have been investigated from the standpoint of continuum mechanics, was proposed in 1955 by Lighthill and Whitham (Ligthill, M.J., Whitham, F.R.S.,1995; Nagel, K. et al, 2003). They showed that the methods for describing transport processes in continuous media can be applied to models of traffic.

In the 60-70s of the 20th century, interest in the study of transport systems was renewed. This interest manifested itself in funding of numerous contracts for the appropriate research topics with the participation of outstanding scientists in the field of mathematics, physics, theory of control, such as I. Prigogine (Prigogine, I.,1961; Prigogine, I., Herman, R., 1971), M. Athans, author of fundamental works on statistics L. Breiman and others. Solution of this problem attracts the best "physical minds".

Today there is an extensive literature on the study and modeling of vehicular traffic. Several academic journals are devoted exclusively to the dynamics of vehicular traffic.

There are two main approaches to modeling the traffic intensity: deterministic and probabilistic models.

Deterministic models are created on the basis of the functional relationships between different indicators, such as speed and distance between vehicles in the stream. Traffic flow in stochastic models is considered as a random process.

All models of traffic flow can be divided into three classes (Helbing, D., 2001): model-analogues, models of "follow the leader", and probabilistic models.

In the models-analogues the motion of vehicles set is simulated by the help of mathematical tools of hydrodynamics. This class of models is called macroscopic.

The models of "follow the leader" are based on the essential assumption of a connection between the movement of the following cars and the head car. According to this approach regularities of motion of the set of vehicles as whole are determined by the local parameters as reaction times of isolated drivers, in particular. This class of models is called microscopic.

In stochastic models the concerted motions of vehicles are considered as a result of stochastic interaction of vehicles on the elements of the transport network. The specific traffic pattern is formed due to the hard nature of the traffic network and by mass character of the movement.

This pattern is characterized by distinct queues and intervals of downloads of the road bands, etc. The appropriate regularities are essentially stochastic.

Recently, in studies of traffic the researchers began to apply interdisciplinary mathematical ideas, methods and algorithms of nonlinear dynamics. Their usefulness is justified by the

presence in the traffic flow of stable and unstable modes of motion, instabilities when changing traffic conditions, and nonlinear feedbacks.
In our opinion, the hydrodynamic models of traffic flow are sufficiently well-grounded. Indeed, traffic flow can be written as a one-dimensional compressible fluid flow, assuming that the consumption is conserved and there is a one-one relation between the speed and the density of traffic.
The first assumption is expressed by the continuity equation. The second assumption is a functional relation between speed and density to account for the reduction of vehicle speed with increasing flux density. This is intuitively correct assumption but theoretically it could lead to the negative of the density or velocity. The point is that, the certain value of the traffic density may correspond to several values of the speed of vehicles.
Therefore, for the correctness of the second assumption the average flow rate at any time ought to meet the equilibrium value for a given density of cars on the road. However, the equilibrium situation is a purely theoretical assumption and can be observed in areas without crossing roads only. On these reasons, some researchers abandoned the continuous models, and some consider them to be too rough.
So it makes sense to continue the search for new models, which would combine the advantages of the microscopic and macroscopic models, and have minimal risks to repeat the shortcomings of known approaches.

2 SEDIMENT MODEL OF VEHICULAR TRAFFIC

2.1 Preliminary notes

As we already noted, there are two main different concepts to describing the traffic, namely: on the base of microscopic or macroscopic approaches. And there are several different types of mathematical models to describe the dynamic evolution of transport systems within the conceptual framework of the microscopic approach.
For example, the individual vehicles are modeled explicitly as particles which move in the environment. Incidentally, under the probabilistic description of traffic flow, the methods developed by the kinetic theory of gases can be used.
Under the deterministic description of the movements of individual vehicles, the simulation is carried out on the principles of the classical Newtonian dynamics.
Computer models are useful to get a better description of the various phenomena characteristic for the traffic, and to understand the basic principles that govern the motion system. Since the large-scale experiments with a real street traffic are not practicable, the use of simulation models in this area of research has no alternative. In this case, empirical data is collected, as a rule, by the passive observation, in contrast to the controlled active experiments used in other fields of study. The use of models and methods of computer simulation helps us to find a strategy for optimizing the transport network and to get the correct methods of traffic control.
Vehicular traffic in the modern metropolis has a very complex structure. Therefore the many usual models which ignore the influence of traffic lights on the vehicles flow can not describe adequately the dynamics of traffic density on the streets, and they can not make a

proper evaluation of the air pollution by exhaust gases, as the pollution and the amount of emissions depend on the work regime of the engine.

In this paper we present a modern heuristic approach to modeling the intensity of vehicular traffic.

2.2 Mathematical details

We submit now the model of vehicular traffic based on the analogy with the sedimentation process (Figure 1).

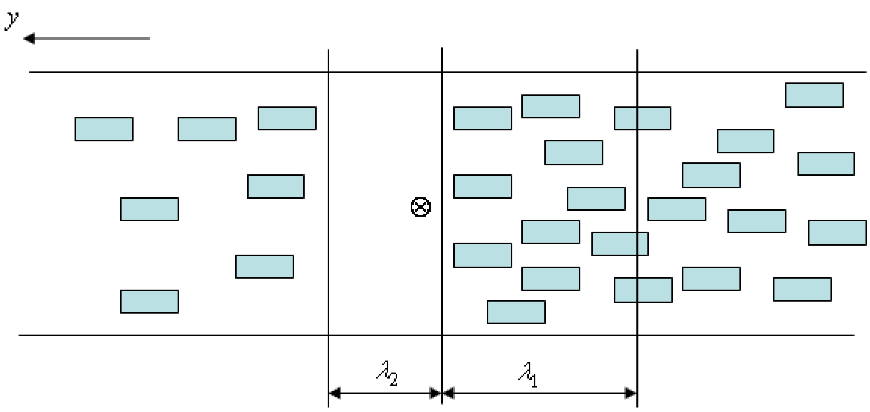

Figure 1- Scheme of traffic flow

Let $V(\rho)$ be the transport rate, depending on the density of vehicles on a road. We distinguish the three characteristic densities, namely: the optimal density ρ_{opt}, under which the maximum allowable speed in urban traffic can be realized, the density of vehicles in traffic in the area nearby traffic lights ρ_{st}, when the red light is on, and the density in a jam ρ_j at which the motion is stopped (Kerner, B.S., Rehborn, H.,1996).

Since due to the influence of traffic lights the density of vehicles on a road depends on the coordinates and time, then from the balance of the number of cars it follows:

$$V\frac{\partial \rho}{\partial y}+\rho\frac{\partial V}{\partial y}=\frac{\partial \rho}{\partial t}, \tag{1}$$

$$\frac{\partial \rho}{\partial t}=-\frac{\partial J}{\partial y}, \tag{2}$$

where J - the traffic flow.

We offer to use the logistic equation to describe the dependence of traffic rate on the traffic density:

$$\frac{\partial V}{\partial \rho}=-kV(V-V_*). \tag{3}$$

Thus, we get:

$$V = \frac{CV_* \exp(-kV_*\rho)}{1 - C\exp(-kV_*\rho)} . \qquad (4)$$

In formula (4) the value of the characteristic $V(\rho_{opt}) = V_{opt}$ velocity V_* is taken for the flux density ρ_{st} , and the constant C is found from the

$$V(\rho_{opt}) = V_{opt} . \qquad (5)$$

Approximately, we can put

$$C \approx \frac{V_{opt}}{V_{opt} + V_*} . \qquad (6)$$

As a result the dependence of the flow rate on the local density reads:

$$V = \frac{V_{opt} V_* \exp(-kV_*\rho)}{V_{opt}(1 - \exp(-kV_*\rho)) + V_*} . \qquad (7)$$

Figure 2 depicts a typical plot of (7).

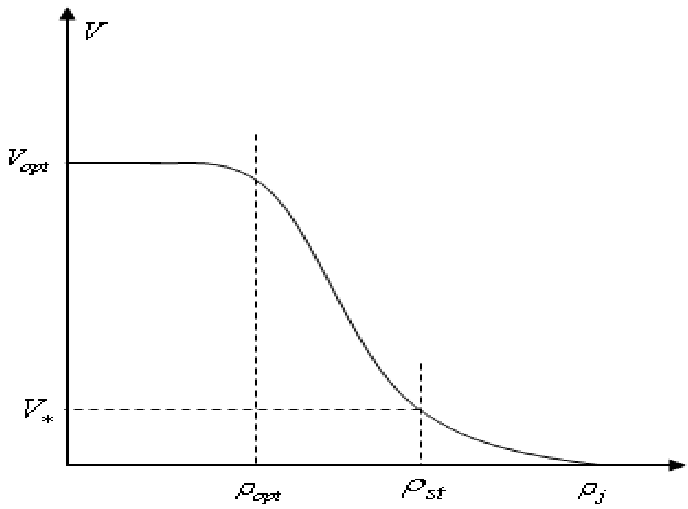

Figure 2- The characteristic dependence of the velocity on the density of traffic flow

Considering equations (1), (2), (7) jointly, we can describe the changes in density of vehicles and traffic intensity over time, and link these changes with the influence of traffic lights, and find also the specific areas of high and low flux densities before and after the traffic light. Substituting (7) in (1), we obtain:

$$V\frac{\partial\rho}{\partial y}+\rho\frac{dV}{d\rho}\frac{\partial\rho}{\partial y}=\frac{\partial\rho}{\partial t}\,. \tag{8}$$

From this it follows

$$\left(V(\rho)+\rho V'\right)\frac{\partial\rho}{\partial y}-\frac{\partial\rho}{\partial t}=0\,, \tag{9}$$

$$V'(\rho)=-\frac{kV_{opt}V_*^2\left(V_{opt}+V_*\right)\exp(-kV_*\rho)}{\left[\left(V_{opt}+V_*\right)-V_{opt}\exp(-kV_*\rho)\right]^2}\,. \tag{10}$$

Full integral of equation (9) can be found by using the Lagrange-Sharpe method.
The scheme of this method in our case is as follows.
Let us consider the system of ordinary differential equations:

$$\frac{dy}{P}=-dt=\frac{d\rho}{Pp-q}=-\frac{dp}{Rp}=-\frac{dq}{Rq}\,, \tag{11}$$

where

$$P=V(\rho)+\rho V'(\rho)\,,$$

$$R=\left(2V'(\rho)+\rho V''(\rho)\right)p\,,$$

$$p=\frac{\partial\rho}{\partial y};\ q=\frac{\partial\rho}{\partial t}\,.$$

The first integral of system (11) reads:

$$\Phi(\rho,p,q)=A=const\,. \tag{12}$$

So we obtain coupled system of equations (9) and (12):

$$\begin{cases}\left(V(\rho)+\rho V'(\rho)\right)p-q=0,\\ \Phi(\rho,p,q)=0\end{cases} \tag{13}$$

Resolving (13) regarding the partial derivatives we can receive the system of "uncoupled" equations:

$$\begin{cases}p=F_1(\rho,A),\\ q=F_2(\rho,A)\end{cases}. \tag{14}$$

Integrals of system (14) will describe the density waves propagating in the traffic flow.
From system (11) it follows:

$$\frac{dp}{dq} = \frac{p}{q}. \tag{15}$$

Therefore, solutions of the model obey the relation

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial y} = 0, \tag{16}$$

where c - the phase velocity of the density wave.
Thus, the model has a solution in the form of a solitary density wave of the form

$$\rho = \rho(\xi), \tag{17}$$

where $\xi = y - ct$ - the automodel wave variable.

The conclusion about that there are density waves on urban motorways is entirely consistent with well-known observations and empirical data (Chowdhury, D. et al, 2000; Lubashevsky, I. et al, 2002).

From (9), (10), (11) the equation for calculating the phase velocity at the front of the traffic wave follows immediately:

$$-c = V(\rho) + \rho V'(\rho). \tag{18}$$

And, as a result, we obtain

$$c = -V\left[1 - \rho k V \frac{V_{opt} + V_*}{V_{opt}\exp(-kV_*\rho)}\right]. \tag{19}$$

When specifying a certain fixed phase velocity, we obtain from (18, 19), the expression for the velocity of vehicles, depending on the density of vehicular traffic at the front of the density wave

$$V = \frac{1 + \sqrt{1 + 4ckA}}{2\rho kA}, \tag{20}$$

where

$$A = \frac{V_{opt} + V_*}{V_{opt}\exp(-kV_*\rho)}. \tag{21}$$

Let us consider the dependence of the total vehicles flow on the road on the density.
From the general expression for the velocity, we obtain

$$J = \frac{\rho V_{opt} V_* \exp(-kV_*\rho)}{V_{opt}(1-\exp(-kV_*\rho)) + V_*}. \qquad (22)$$

Figure 3 depicts the typical plots of the total vehicular traffic, calculated from equation (22) at different values V_*. Figure 4 depicts the experimental data corresponding to observations made on Canadian highways (Lubashevsky, I. et al, 2002; Kerner, B.S., 2004).

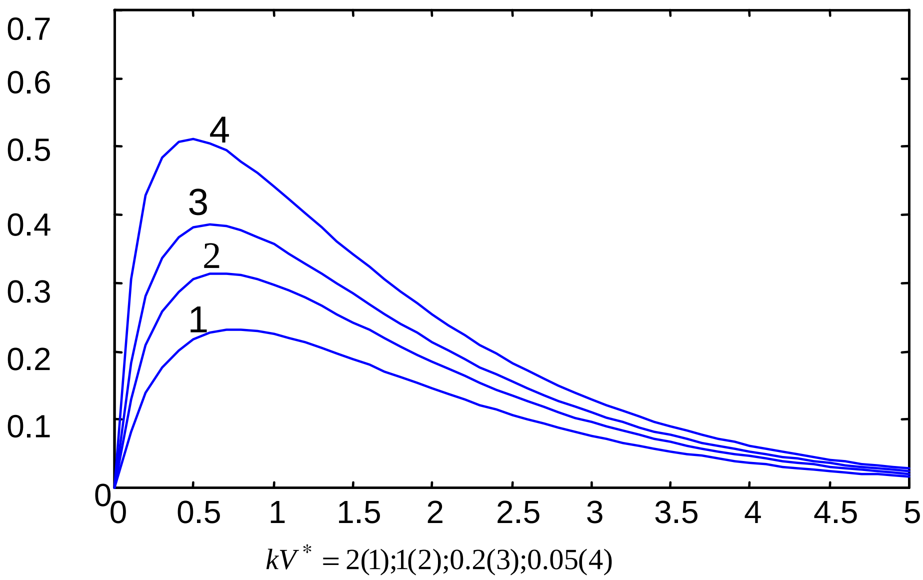

Figure 3- Typical plots of the total traffic flow J (1/(ms)) on the density of traffic ρ (1/m)

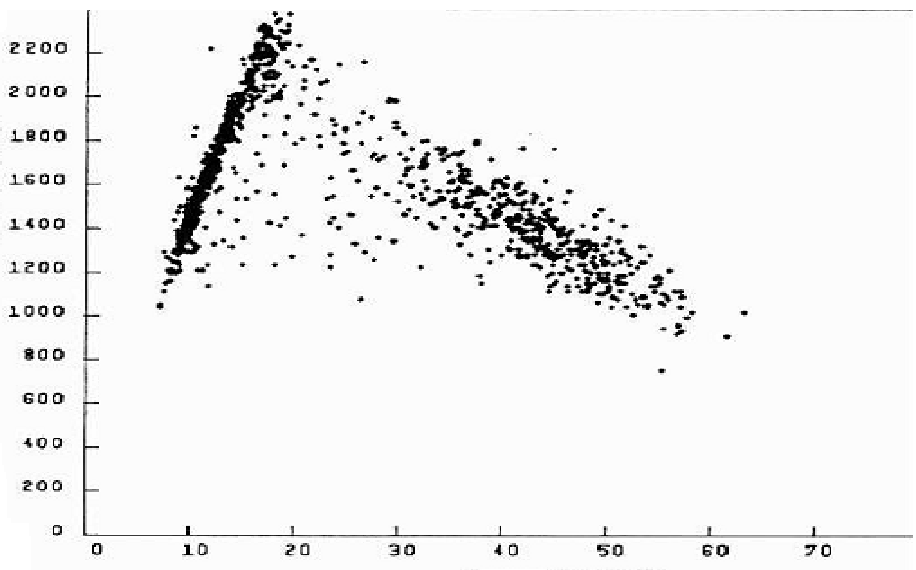

Figure 4- Experimental data on the flow (1/h) and the relative traffic density (%) on Canadian highways (Lubashevsky, I. et al, 2002).

It is easily to be convinced of that submitted model gives true qualitative description of the phenomenon.

2.3 Assessment of the critical density of vehicular traffic and pre-jam situations

Now we will get the evaluation for the value of the critical density, at which there is maximum of the vehicular traffic intensity on straight stretches of road without intersections and crossings.
This critical value should satisfy:

$$\left(V_{opt} + V_*\right)\left(1 - kV_*\rho_{\kappa p}\right) = V_{opt}\exp\left(-kV_*\rho_{\kappa p}\right). \tag{23}$$

Figure 5 is an illustration of relation (23) that can be interpreted as a transcendental equation for the calculation of the critical density, after which the rapid decline in traffic intensity may be observed.
Since the reduction of the total flux begins in the area of strong dependence of the motion velocity on the flux density, we can propose the approximate explicit formula for the calculation of the critical density.
To do this, we expand the exponent in (23) into series and after saving of two terms of the expansion we obtain the following expression:

$$\rho_{\kappa p} \approx \frac{V_* + \sqrt{V_*^2 + 2V_{opt}V_*}}{2V_{opt}V_*}. \tag{24}$$

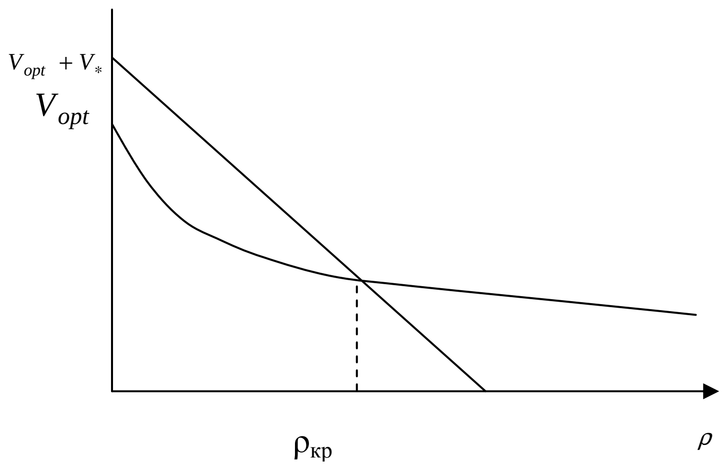

Figure 5- The method of calculating the critical density according to the sediment model (24)

The last formula is easily converted to the form:

$$\rho_{\kappa p} \approx \frac{1 + \sqrt{1 + 2s}}{2V_{opt}}, \tag{25}$$

where $s = V_{opt}/V_*$ - the ratio of the optimum speed of the roadway to the speed of braking.

Figure 5 depicts the results of measurements of traffic flow under decreasing and increasing the relative traffic density. The characteristic hysteresis is noticeable, i.e., reducing of the traffic density does not lead to a significant increase of the traffic flow (Chowdhury, D. et al, 2000; Treiterer J., Myers J.A., 1974).

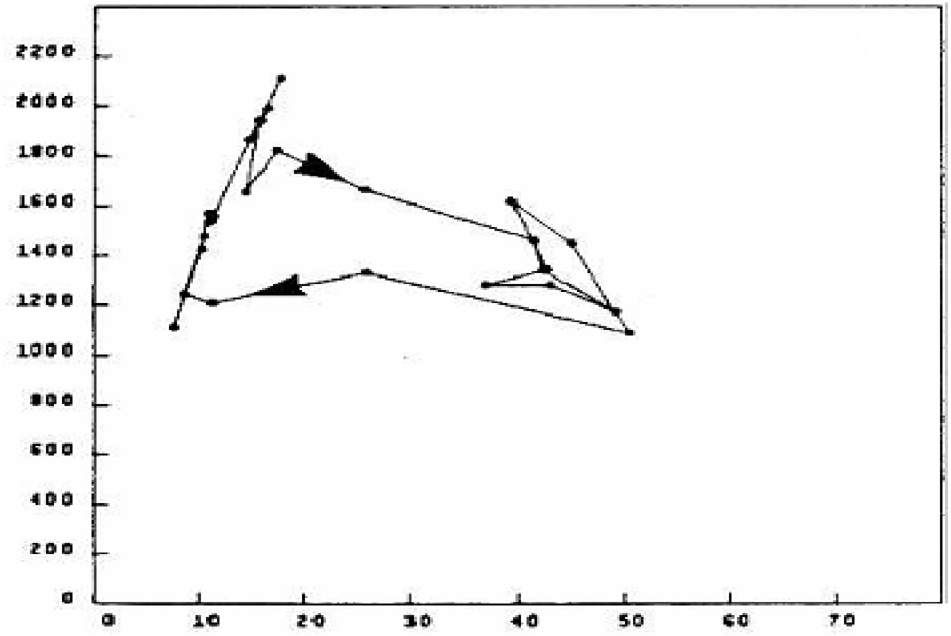

Figure 6- Transport hysteresis according to (Chowdhury, D. et al, 2000)

This phenomenon is also explained by the sediment model.
Indeed, with a decrease in the flux density on the descending branch of the dependence of flow on the density $J(\rho)$ the wave front with the corresponding phase velocity (3.22) is formed. And the speed at the wave front (20), under the great density, is almost inversely proportional to it.
Hence for the total flux under the "reverse course" of the density we obtain:

$$J = \frac{1+\sqrt{1+4ckA}}{2kA}, \tag{26}$$

where

$$A = \frac{V_{opt} + V_*}{V_{opt} \exp(-kV_* \rho)}.$$

When $\rho \to 0$ we have:

$$J_0 \approx \frac{1+\sqrt{1+4ck\left(1+\frac{1}{s}\right)}}{2k\left(1+\frac{1}{s}\right)}. \tag{27}$$

Figure 7 is the illustration of this phenomenon, which according to the sedimentation model can be interpreted as instability of the descending branch of the dependence $J(\rho)$.

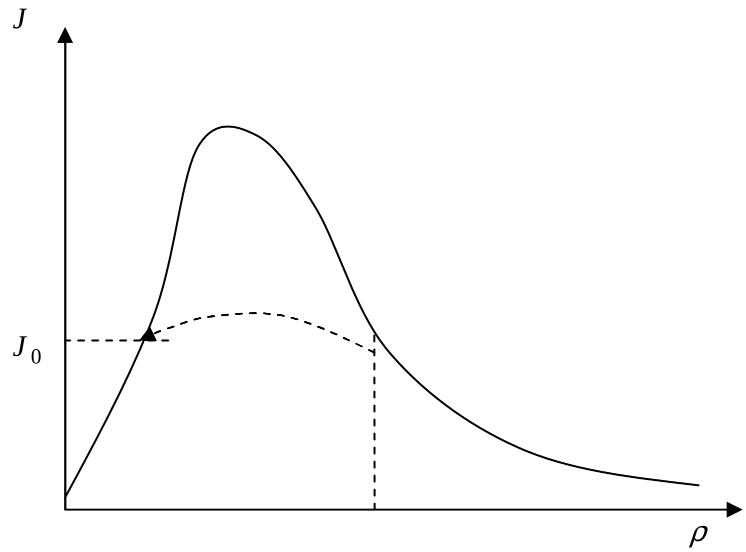

Figure 7- interpreting the transport hysteresis according to the sediment model

Other characteristic traffic problems are associated with the appearance of "traffic jams." Figure 8 shows a plot of the trajectories of individual vehicles on multi-lane highway. These data were obtained with the help of aerial photography.

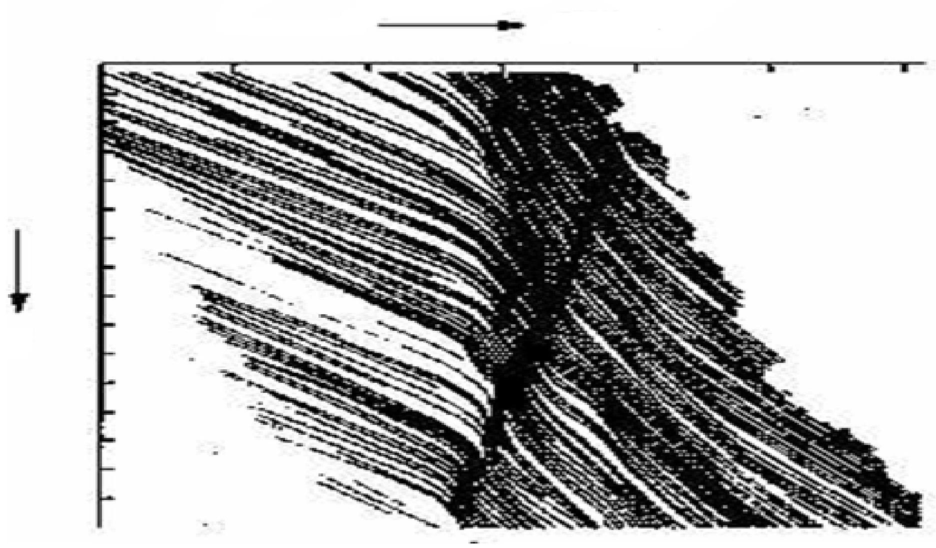

Figure 8- Aerial view of the trajectories of individual vehicles on multilane highway (Lubashevsky, I. et al, 2002)

It is known that phenomena of the spontaneous formation, propagation, and "resolution" of traffic jams on the stretch of road are often observed for no apparent and explicit reason.

2.4 Phenomenon of jam

In our opinion none of the known models can lay claim to an adequate description of the formation of traffic jams.

Here we wish to submit some ideas and simple calculations, which in certain cases may be the basis for models describing the phenomenon of jams.

Suppose that a set of vehicles with the traffic density ρ moves along a single-lane highway with a given width h and with a length L of unimpeded flow of traffic (Figure 9).

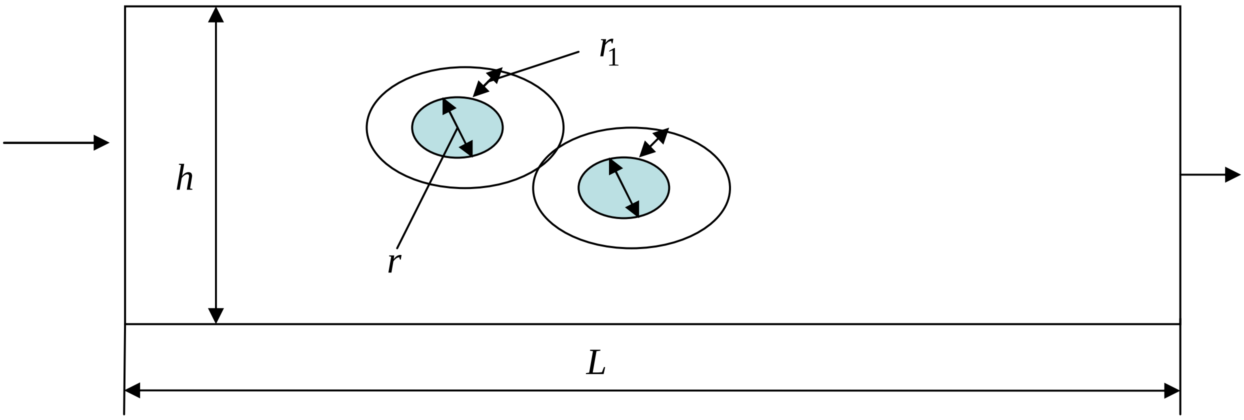

Figure 9- To deriving condition (35)

The residence time of vehicles with average speed V at the given site is

$$T = \frac{L}{V} \ . \tag{28}$$

The number of cars arriving per the time unit to this site is:

$$q = \rho h \ . \tag{29}$$

Thus, the following number of cars is on the highway simultaneously.

$$N = \frac{\rho h L}{V} \, . \tag{30}$$

Let us estimate the area which is occupied by every car on the road as

$$f = \psi (r + r_1)^2 \, , \tag{31}$$

where

r is the characteristic size of a single vehicle;

r_1 is the characteristic size of the car maneuver;

ψ is the factor of the form of the car.

From relations (30) and (31) it follows

$$\frac{\rho hL}{V} = \frac{hL}{\psi(r + r_1)^2}, \qquad \text{or } r_1 = \sqrt{\frac{V}{\psi\rho}} - r \; . \tag{32}$$

Our main hypothesis is that the characteristic radius of the maneuver can not be less than a certain value that is determined by the time of driver's response to disturbances which can arise during movement.
In accordance with this hypothesis the size of maneuver at a given speed should satisfy

$$r_1 = \sqrt{\frac{V}{\psi\rho}} - r \geq V\tau \; . \tag{33}$$

Hence we obtain

$$\frac{V}{\psi\rho} \geq (r + V\tau)^2 \, . \tag{34}$$

From the existence of solutions of inequality (34) regarding the velocity, we obtain

$$\rho \leq \frac{1}{4\psi r\tau} \, . \tag{35}$$

The motion becomes impossible under the condition.

$$\rho > \rho_{\kappa p} = \frac{1}{4\psi r\tau} \, . \tag{36}$$

From this it follows that the critical speed corresponding to the "pre-jam" state is defined by the relation

$$V_{\kappa p} = \frac{r}{\tau} \, . \tag{37}$$

In other words, a jam occurs when the vehicle speed corresponding to a given density becomes equal to the speed of maneuver at the length which is equal to the characteristic size of the car.

3 CONCLUSION

The new sediment model of vehicular traffic has been submitted. It is shown that the use of the logistic equation to describe the dependence of the density of traffic flow on its intensity allows to give an interpretation of the known experimental data of vehicles traffic on the highway. It is also shown that the developed model explains the known phenomenon of the hysteresis of traffic intensity depending on the density of traffic flow. The new model for prediction of the effects of jams on the roads has been presented too. The model is based on the idea of evaluation of the characteristic maneuver time while driving the vehicle in the stream.

The offered model and appropriate equations can be used at designing of highways, reconstruction of roads and highways, for exact regulations of traffic lights, and also for working out actions for preventing the jams on roads which are one of principal causes of air pollutions and discomfort of inhabitants of cities.

REFERENCES

Cremer, M., Ludwig, J. (1986). A fast simulation model for traffic flow on the basis of Boolean operations. Math. Comp Simul., 28, 297-303.

Chowdhury, D., Santen, L., Schadschneider, A. (2000). Statistical physics of vehicular traffic and some related systems. Phys. Rep., 329, 199-329.

Greenshields, B.D. (1934). A study of traffic capacity. Proc. (US) highway research. board., 14, 448--494.

Gasis, D.C. et al. (1959). Car following theory of steady state flow. Operations Research.,7, 499-505.

Greenberg, H. (1959). An analysis of traffic flow. Operations Research., 7 , 79--85.

Haight, F.A. (1963). Mathematical theories of traffic flow. Academic press, New York, London.

Helbing, D. (2001),Traffic and related self-driven many-particle systems, Rev. Mod. Phys., 73, 1067.- hbc2

Kerner, B.S., Rehborn, H. (1996). Experimental Features and characteristics of traffic jams. Physical Review E., 53, 1297-1300.

Kerner, B.S. (2004). Physics of Traffic. Springer, Berlin.

Ligthill, M.J., Whitham, F.R.S. (1995). On kinetic waves II. A theory of traffic flow on crowded roads. Proc. of the Royal Society Ser. A., 229, No. 1178, 317-345.

Lubashevsky, I., Mahnke, R., Wagner, P., and Kalenkov, S. (2002). Long-lived states in synchronized traffic flow: Empirical prompt and dynamical trap model. Phys. Rev. E., 66, 016117. hbc1

Nagel, K., Wagner, R., Woesler, R. (2003). Still flowing. Approaches to traffic flow and traffic jam modeling, January 2.

Prigogine, I. (1961). A Boltsman-like approach to the statistical theory of traffic flow /Theory of Traffic Flow. Ed. Herman R. Elsevier, Amsterdam.

Prigogine, I., Herman, R. (1971). Kinetic Theory of Vichicullar Traffic. Elsevier, N.Y.

Treiterer J., Myers J.A. (1974) The hysteresis phenomenon in tfaffic flow. In D.J. Buckley, editor, Proc. 6th ISTT, 13, Artarmon, New South Wales.