A DISAGGREGATE RESIDENTIAL EQUILIBRIUM ASSIGNMENT MODEL

Leurent Fabien, Université Paris Est, Laboratoire Ville Mobilité Transport, Ecole des Ponts ParisTech

Coulombel Nicolas, Université Paris Est, Laboratoire Ville Mobilité Transport, Ecole des Ponts ParisTech. nicolas.coulombel@enpc.fr

Poulhès Alexis, Université Paris Est, Laboratoire Ville Mobilité Transport, Ecole des Ponts ParisTech. alexis.poulhes@enpc.fr

ABSTRACT

This paper presents an equilibrium model for urban housing markets, with an application to the Paris metropolitan area. The model focuses on two elements: a correct representation of the various market segments, with a particular attention to the location, size and quality of the dwelling, and the role of household income. Housing supply is assumed to be price-elastic in a way that can vary for each location and each housing type. Regarding housing demand, households are stratified according to the household size and the workplace and the occupation (including job type) of the household head. Each household class is further characterized by an income distribution, income being treated as a continuous variable (and not a discrete one as done in most applied models). Market equilibrium is characterized first by a set of primal-dual conditions, then by a variational inequality the continuity of which guarantees the existence of an equilibrium. In the monocentric case, primal-dual conditions are the saddle point conditions of a constrained maximization program. Concavity of this program implies the uniqueness of the equilibrium. A resolution algorithm is derived from traffic assignment on a transport network. We then present preliminary results of an application to the Paris region, including a test scenario of increased fuel prices, as a way to illustrate the potentialities of the model.

Keywords: housing market, disaggregate demand, supply and demand equilibrium, income distribution, variational inequality.

INTRODUCTION

Context

Two socio-economic systems are of particular interest for urban planning: the land use system (which comprehends the real estate system) and the transportation system. These are strongly intertwined, as established by a now considerably large body of literature (e.g. Levinson 2007, and King 2011 for recent contributions, and Lowry, 1964, for an ancient one). The relationship between the two is not symmetrical however, with urban change processes working on different time scales (Simmonds *et al*, 2011). Land Use – Transport Interaction (LUTI) models, which very aim is to model these urban change processes linking those two systems, are therefore especially helpful for the design and ex-ante evaluation of urban development scenarios.
The typology proposed by David Simmonds Consultancy *et al* (1999) distinguishes two main families of LUTI models: static models such as Relu-Tran (Anas and Liu, 2007) or Pirandello (Delons *et al*, 2009), and which focus on the long term supply – demand equilibrium: versus quasi – dynamic models such as UrbanSim (Waddell et al., 2003), or Ilute (Salvini and Miller, 2005), and which analyze the dynamics of the two systems. The former are the simpler and are therefore the natural choice in order to simulate an area and assist the planning process – before using a more detailed and more complex dynamic model if this proves necessary.
It is above all vital for the model's hypotheses to be realistic and to be verifiable by an analyst. Most static models assume that the total floor area of the dwellings in each zone in the area can be influenced by the demands of households and their accumulation, and also that each demander is immutable ([1]). The dynamic framework removes these limitations, but at the cost of an increase in the number of hypotheses whose interactions are not only more difficult to check but even to identify qualitatively.

Objective

This paper presents a static model for the equilibrium between supply and demand on an urban housing market. This model disaggregates demand according to place of work, category of activity and household size and income. It is designed for the private rented property market, so social housing and home purchase are not dealt with at this stage but may be covered by subsequent extensions. The size of each dwelling is fixed, and the number of dwellings, broken down on a qualitative basis (which includes size) and with respect to location, is elastic to price according to a given specification.
We shall provide an efficient mathematical formulation for this model: starting from primal-dual equilibrium conditions, a variational inequality problem will be formulated; a concave maximization program is available also for a monocentric instance. This has enabled us to demonstrate the existence and uniqueness of the equilibrium and propose a robust computation method.

[1] i.e. which does not change over time, without the concept of life-cycle changes.

The model is intended for simple applications, providing an approximate result in the context of an operational study or for teaching purposes.

The approach

Overall, an economic model is specified for which a mathematical and algorithmic treatment is provided: the paper is situated at the crossroads between urban economics and operations research.
The mathematical treatment is derived from traffic assignment on a transport network, in particular, dual criteria assignment with a cost-time trade-off. Demand will be disaggregated discretely according to the employment area and household size, and continuously according to income. For each household, we will model the discrete choice between the housing options, according to zone and quality (which includes size), assuming that the household is sensitive to the price of housing and the cost of transport between home and work. Transport between home and work has a monetary cost and a temporal cost which determine budgetary accessibility and affect utility in relation to income. Thus, for each employment area and discrete type of household, the choice of residential zone and quality of dwelling is analogous to the choice of a route on an origin-destination link on the basis of cost and time, for a population of demanders who are differentiated on the basis of the cost-time trade-off. We will adapt our previous treatment of the dual criteria assignment of traffic onto a network (Leurent, 1993) to the microeconomic choice of housing: this adaptation is straightforward for a simple utility function e.g. Cobb-Douglas or similar.

The structure of the paper

The main body of the paper is in four parts followed by a conclusion. We begin by stating the hypotheses of the model (Section 2). From these we derive a "grouped" demand function by aggregating microeconomic agents without restricting their individual behaviour (Section 3). Next, we will formulate the supply-demand equilibrium, give mathematical formulations, establish the properties of existence and uniqueness, and provide a resolution algorithm (Section 4). Last, we present the first results from an application to the Paris region, including the analysis of a test scenario of increased fuel prices. (Section 5). The conclusion will set out some possibilities for extension (Section 6).

MODELLING HYPOTHESES

The model's hypotheses relate to job supply (§ 2.1), housing supply (§ 2.2), transport supply (§ 2.3) and demand for housing and transport (§ 2.4).

Job supply

Let us assume that the number of jobs per zone for each category of activity (which can include the nature of the activity and the job category) and for each level of salary is exogenous in the medium term. Let $W = J \times K$ be the set of pairs $w = (j,k)$ for an employment area $j \in J$ and a category of activity $k \in K$. The number of jobs in the category is denoted by Q_w, and the salaries are distributed according to a distribution function H_w. Thus, economic activity in the area does not depend on the economic conditions that apply to the housing and transport markets.

Housing supply

Let us assume that housing is distributed according to quality and locational zone. Quality includes, in particular, the floor area of the dwelling, the quality of construction, the quality of the neighbourhood and the available local amenities. R denotes all the types of property, s_r denotes the floor area of the type r dwellings, q_r the residual attributes of quality, S_r the quantity supplied and p_r the price per unit of floor area.

Assume also that in the medium term supply is elastic to the market price according to the function

$$S_r = \mathrm{O}_r(p_r). \quad (2.1)$$

By assumption, O_r is an increasing function. O_r^{-1} is the reciprocal function, which is also increasing.

In principle, these hypotheses are valid for the private rented property sector. The social rented property sector operates differently, with a supply function that depends both on the production cost and private sector prices, while demand depends both on the private sector price and the waiting delay. The owner-occupier sector bears a closer resemblance to the private rented sector than the social sector, but the transaction costs depend on the length of occupation and the price for the demander depends on the cost of credit and the initial deposit (therefore capital as well as income).

Transport supply

It is assumed that transport services are available between the different types of residential areas (by zone and housing mode) $i \in I$ and the different types of job $w \in W$ (making a distinction according to the zone and, if necessary, the category of activity which can generate specific modal requirements such as car access). Servicing the pair (i, w) involves a travel time $\tilde{t}_{iw}$ and a monetary cost t_{iw}.

These terms are taken as exogenous, i.e. independent of network flows and the income derived from the job held.

Demand for housing and transport

Household demand is the keystone of the system as it links jobs, housing and transport. We have assumed that demand consists of a set of households, each of which has one job and a size ν_m that is expressed as the number of individuals or consumption units. The household's job is specified in terms of location, category of activity k_m and salary ρ_m.

For the household, a dwelling of type r is a residential option with a floor area s_r and a quality q_r, and carries with it housing expenditure of $p_r.s_r$ and travel expenditure (t_{rj} in money and $\tilde{t}_{rj}$ in time). These expenditures reduce the amount of the household's income that is available for all types of consumption apart from housing and transport. These types of consumption other than housing and transport are known as the "numéraire good" which is monetized and denoted by z.

We have made the assumption that the household desires space in its dwelling, quality (apart from the floor area) denoted by q, the numéraire good z and free time τ, and that its preferences for a "bundle" (s,q,z,τ) are represented by a utility function $\mathrm{U}_m(s,q,z,\tau)$. This function models the household's interest in the bundle. Each housing option constitutes a specific bundle. The household is assumed to be micro-economically rational, meaning that he selects the option with the maximum utility from those on the market which are accessible to him.

The household evaluates a residential option in relation to two budgetary constraints, one of which is monetary and the other temporal. Denote by $c_r(q_r, s_r, \nu_m)$ the running cost of the dwelling in each period with respect to household size, including possibly a reserve price from the supplier. The monetary budgetary constraint is for the household's expenditure on housing, transport and the numéraire good to be compatible with his income:

$$p_r s_r + c_r(q_r, s_r, \nu_m) + t_{rj} + z \leq \rho_m. \quad (2.2)$$

The temporal budgetary constraint compares the time at work $\tilde{\rho}_m$, travel time $\tilde{t}_{rj}$ and the residual time τ to the available time θ_m:

$$\tilde{\rho}_m + \tilde{t}_{rj} + \tau \leq \theta_m. \quad (2.3)$$

Let R_m denote all the housing options which are within the household's budgetary reach, i.e. which satisfy (2.2) and (2.3). An option of this type leaves the household with an amount of the numéraire good and free time that are formulated as follows:

$$\tilde{z}_{mr} = \rho_m - p_r s_r - c_r(q_r, s_r, \nu_m) - t_{rj}, \quad (2.4)$$

$$\tilde{\tau}_{mr} = \theta_m - \tilde{\rho}_m - \tilde{t}_{rj}. \quad (2.5)$$

Its utility for the household is therefore

$$\tilde{\mathrm{U}}_m(r) = \mathrm{U}_m(s_r, q_r, \tilde{z}_{mr}, \tilde{\tau}_{mr}). \quad (2.6)$$

The demander's microeconomic behaviour consists of selecting the best available option, i.e.

$$\text{Finding } r^* \in R_m,\ \forall r \in R_m,\ \tilde{\mathrm{U}}_m(r^*) \geq \tilde{\mathrm{U}}_m(r). \quad (2.7)$$

THE DEMAND FUNCTION FOR EACH CLASS (SEGMENT)

We have specified the microeconomic model for an individual demander of housing and transport. This model is disaggregated on the basis of job location j, category of activity k, income level ρ and household size ν. Disaggregation is performed discretely for j,k,ν and continuously for ρ. A demand class is defined by the triple $\sigma = (j,k,\nu)$ and a distribution function H_σ for income. Income is thus the central parameter of our analysis. By considering its distribution we treat a class collectively in order to formulate its overall demand for a given set of supplied options.

Residual consumption and residential utility

The relationship between the residual amount of the numéraire good and income is affine: denoting $P_{\sigma r} = p_r s_r + c_r(q_r, s_r) + t_{rj}$, equation (2.4) yields that

$$\tilde{z}_{\sigma r} = \rho - P_{\sigma r}, \quad (3.1)$$

which is an increasing function of income.
The amount of free time $\tilde{\tau}_{mr}$ does not depend directly on income and can be denoted by $\tilde{\tau}_{\sigma r}$.
The residential utility $\tilde{\mathrm{U}}_m(r)$ of a residential option r for a demander $m = (\sigma, \rho)$ is the utility function $\mathrm{U}_m(s_r, q_r, \tilde{z}_{mr}, \tilde{\tau}_{mr})$, on condition the option is compatible with the temporal and monetary budgetary constraints. This compatibility depends simultaneously on r, σ and ρ. When this compatibility exists (each constraint must be examined in relation to ρ), under given r and σ, the residential utility is an increasing function of the residual consumption $\tilde{z}_{\sigma r}$, and therefore increases with income as $\tilde{z}_{\sigma r}$ increases with ρ.
Let $\hat{\mathrm{U}}_{r\sigma}(\rho)$ denote the utility of the option r in relation to income ρ, within the segment σ. This function is increasing and continuous if the initial function U_m is continuous.
Let us also define the maximum residential utility function out of all the budgetarily feasible options,

$$\mathrm{U}^*_{R\sigma}(\rho) = \max\{\hat{\mathrm{U}}_{r\sigma}(\rho) : r \in R_{\sigma\rho}\}. \quad (3.2)$$

As an increase in income makes it easier to satisfy a budgetary constraint, a higher income makes it possible to access more residential options. A higher income therefore provides higher maximum residential utility for two reasons: first because the utility of each accessible option is higher and second because the choice set is larger.

Efficiency domain of a housing option

For a given option r, we shall define the feasibility domain $\mathrm{F}_{\sigma r} = \{\rho : r \in R_{\sigma\rho}\}$ and the efficiency domain, which is the set of incomes for which the option provides the maximum utility:

$$\mathrm{E}_{\sigma r} = \{\rho : r \in R_{\sigma\rho} \text{ and } \hat{\mathrm{U}}_{r\sigma}(\rho) = \mathrm{U}^*_{R\sigma}(\rho)\}. \quad (3.3)$$

The assignment of each demander to their preferred option amounts to specify the efficiency domains of the various options. If the utility functions have a simple form the efficiency domains do too, as they are intervals that are bounded by special values, the break-off points. Let us make the specific hypothesis that there is an increasing function φ_σ such that, for every option, the compound function $\mathrm{u}_{\sigma r} = \varphi_\sigma \circ \hat{\mathrm{U}}_{\sigma r}$ is affine in relation to income. As φ_σ is an increasing function, so is $\rho \mapsto \mathrm{u}_{\sigma r}(\rho)$, which is a utility function that depends ultimately on s_r, q_r, $\tilde{z}_{\sigma r}$ and $\tilde{\tau}_{\sigma r}$. Let us define the affinity parameters a and b according to income such that

$$\mathrm{u}_{\sigma r}(\rho) = a_{\sigma r} + b_{\sigma r}\rho = b_{\sigma r}(\rho - \xi_{\sigma r}). \quad (3.4)$$

The efficiency domain of an option is the intersection between its relative efficiency domains compared to any other option: $\mathrm{E}_{\sigma r} = \bigcap_{s \in R\rho} \mathrm{E}_{\sigma r/s}$, for the relative domain

$$\mathrm{E}_{\sigma r/s} = \{\rho \in \mathrm{F}_{\sigma r} : \mathrm{u}_{\sigma r}(\rho) \geq \mathrm{u}_{\sigma s}(\rho) \text{ if } \rho \in \mathrm{F}_{\sigma s}\}. \quad (3.5)$$

Under the hypothesis of affinity, we can re-express the condition of superiority thus:

$$\mathrm{u}_{\sigma r}(\rho) \geq \mathrm{u}_{\sigma s}(\rho) \Leftrightarrow b_{\sigma r}(\rho - \xi_{\sigma r}) \geq b_{\sigma s}(\rho - \xi_{\sigma s})$$
$$\Leftrightarrow (b_{\sigma r} - b_{\sigma s})\rho \geq b_{\sigma r}\xi_{\sigma r} - b_{\sigma s}\xi_{\sigma s}$$

If $b_{\sigma r} \neq b_{\sigma s}$, both sides of the previous line are equal at $\rho^*_{rs} = [b_{\sigma r}\xi_{\sigma r} - b_{\sigma s}\xi_{\sigma s}]/(b_{\sigma r} - b_{\sigma s})$, and

$\mathrm{u}_{\sigma r}(\rho) - \mathrm{u}_{\sigma s}(\rho) = (b_{\sigma r} - b_{\sigma s}).(\rho - \rho^*_{rs})$.

So, if $b_{\sigma r} > b_{\sigma s}$, r is preferred when $\rho \geq \rho^*_{rs}$ and s when $\rho \leq \rho^*_{rs}$. If $b_{\sigma r} < b_{\sigma s}$, s is better above ρ^*_{rs} and r is better below. In addition, it is necessary to include the admissibility conditions for each option: for r the financial constraint is $\tilde{z}_{\sigma r} \geq 0$ so $\rho \geq P_{\sigma r}$ and the temporal constraint is $\tilde{\tau}_{mr} \geq 0$ so $\theta_m - \tilde{\rho}_m \geq \tilde{t}_{rj}$.

Finally, denoting by $\mathrm{D}_{\sigma,r/s}$ the efficiency interval in the absence of admissibility constraints, it holds that

$$\mathrm{E}_{\sigma r/s} = \mathrm{F}_{\sigma r} \setminus (\mathrm{D}_{\sigma,r/s} \cap \mathrm{F}_{\sigma s}) = \mathrm{D}_{\sigma,r/s} \cap \mathrm{F}_{\sigma s} \cap \mathrm{F}_{\sigma r} \setminus (\mathrm{F}_{\sigma r} \setminus \mathrm{F}_{\sigma s}). \qquad (3.6)$$

The efficiency domains $\mathrm{E}_{\sigma r}$ of the options can be determined in a comprehensive manner, integrating the options in the order of increasing values of $b_{\sigma r}$. Including a new option conserves or reduces the domain of each of the previous options, but it remains an interval.

Ultimately, $\mathrm{E}_{\sigma r} =]\underline{\rho}_{\sigma r}, \overline{\rho}_{\sigma r}[$, perhaps with closed brackets and/or $\overline{\rho}_{\sigma r} = +\infty$. $\mathrm{E}_{\sigma r}$ may be empty, which we consider to be equivalent to $\underline{\rho}_{\sigma r} = \overline{\rho}_{\sigma r}$ if the distribution H_σ is continuous. The intervals are arranged in increasing order of $b_{\sigma r}$, so between two consecutive non-punctual intervals $s < r$, $\overline{\rho}_{\sigma s} = \underline{\rho}_{\sigma r} = \rho^*_{sr}$.

The demand function for each class

Under these conditions, the option r is optimum for $\mathrm{E}_{\sigma r}$ and selected for every value of $\rho \in \mathrm{E}_{\sigma r}$ (save possibly for the break-off points). Therefore it attracts customers from segment σ in number of

$$n_{\sigma r} = Q_\sigma.\Pr\{\rho \in \mathrm{E}_{\sigma r}\} = Q_\sigma[\mathrm{H}_\sigma(\overline{\rho}_{\sigma r}) - \mathrm{H}_\sigma(\underline{\rho}_{\sigma r})]. \quad (3.7)$$

The $[n_{\sigma r} : r \in R_\sigma]$ make up the demand function of segment σ for the residential options. They depend on the conditions of supply.

Quasi Cobb-Douglas utility function

The framework of hypotheses is particularly appropriate for the following utility function:

$$\mathrm{U}_\sigma(s, q, z, \tau) = \mathrm{K}_\sigma((s - \nu_\sigma s_0)^+)^\alpha q^\beta z^\gamma \tau^\delta, \qquad (3.8)$$

which is of the Cobb-Douglas type with the parameters $(\alpha, \beta, \gamma, \delta)$ which may themselves depend on σ, and that is modified by requiring a minimum floor area, denoted by s_0, for each household member. The utility of an option whose floor area is too small is reduced to zero.

The function $\varphi_\sigma : x \mapsto x^{1/\gamma}$, applied to U_σ, gives the following affine function:

$$\mathrm{u}_{\sigma r}(\rho) = [\mathrm{K}_\sigma((s - \nu_\sigma s_0)^+)^\alpha q^\beta \tilde{\tau}^\delta]^{1/\gamma}\, \tilde{z}_{\sigma r}(\rho) = b_{\sigma r}(\rho - P_{\sigma r}). \, (3.9)$$

Therefore $b_{\sigma r} = [\mathrm{K}_\sigma((s-\nu_\sigma s_0)^+)^\alpha q^\beta \tilde{\tau}^\delta]^{1/\gamma}$ and $\xi_{\sigma r} = P_{\sigma r}$ and $a_{\sigma r} = -b_{\sigma r}\xi_{\sigma r} = -b_{\sigma r}P_{\sigma r}$.
The order of the $b_{\sigma r}$ values depends on the characteristics q_r and s_r as well as the travel time $\tilde{t}_{rj}$ between the home r and zone of work j for segment σ, via $\tilde{\tau}_{\sigma r} = \theta_m - \tilde{\rho}_m - \tilde{t}_{rj}$.
A longer travel time reduces the $b_{\sigma r}$ and therefore acts in the opposite direction to a higher floor area s_r. All other things being equal, options which are identical in all ways except for travel time are arranged in order of decreasing travel time so, a priori, the efficiency domain of high incomes favours short home-to-work distances.

MATHEMATICAL ANALYSIS

Primal-dual equilibrium conditions

The option r and the segment σ we shall associate the sum $N_{\sigma r} = \sum_{\ell=1}^{r} n_{\sigma\ell}$ and the attractiveness function

$$\mathrm{I}_{\sigma r} = a_{\sigma r} + \sum_{\ell=r}^{R_\sigma - 1}(b_{\sigma\ell} - b_{\sigma,\ell+1})\,\mathrm{H}_\sigma^{-1}(N_{\sigma\ell}/Q_\sigma). \qquad (4.1)$$

Let us assume that each demander is assigned to their optimum option. Then, between two options s and r that are used consecutively,

$$\mathrm{I}_{\sigma r} - \mathrm{I}_{\sigma s} = a_{\sigma r} - a_{\sigma s} - \sum_{\ell=s}^{r-1}(b_{\sigma\ell} - b_{\sigma,\ell+1})\,\mathrm{H}_\sigma^{-1}(N_{\sigma\ell}/Q_\sigma)$$
$$= a_{\sigma r} - a_{\sigma s} + (b_{\sigma r} - b_{\sigma s})\rho^*_{sr}$$
$$= 0 \text{ from the definition of } \rho^*_{sr}$$

For an unused option t, the values $\underline{\rho}_{\sigma t}$ and $\overline{\rho}_{\sigma t}$ are fixed at the break-off value between the two used options s and r which are respectively lower and higher in the order of increasing values of $b_{\sigma r}$, so

$$\mathrm{I}_{\sigma r} - \mathrm{I}_{\sigma t} = a_{\sigma r} - a_{\sigma t} + (b_{\sigma r} - b_{\sigma t})\rho^*_{sr} \geq 0 \text{ since } \mathrm{u}_{\sigma r}(\rho^*_{sr}) \geq \mathrm{u}_{\sigma t}(\rho^*_{sr}).$$

Thus, the attractiveness function is at a maximum for any option that is used, and an option with a value that is strictly lower than the maximum value cannot be efficient (i.e. have a non-trivial efficiency domain). We can therefore define the equilibrium conditions with a dual variable μ_σ as follows:

$$n_{\sigma r} \geq 0 \text{ and } \sum_{r=1}^{R_\sigma} n_{r\sigma} = Q_\sigma \qquad (4.2\text{a,b})$$
$$\mathrm{I}_{\sigma r} - \mu_\sigma \leq 0 \text{ and } n_{\sigma r}.(\mathrm{I}_{\sigma r} - \mu_\sigma) = 0 \qquad (4.2\text{c,d})$$

It can easily be shown that these conditions are sufficient to characterize a local equilibrium for the segment σ. Coordination between segments is achieved by adjusting the price and quantity of the supply: $a_{\sigma r}$ depends on the price p_r which in turn depends on $S_r = \sum_\sigma n_{\sigma r}$. This provides the basis for the general characterization that follows.

Variational inequality and property of existence

In vectorial terms, let us denote $\mathbf{n}_\sigma = [n_{\sigma r} : r \in R_\sigma]$ and $\mathbf{I}_\sigma = [\mathrm{I}_{\sigma r} : r \in R_\sigma]$ with components that are arranged in order of increasing $b_{\sigma r}$ values.

***Theorem**. The residential equilibrium resolves the following variational inequality problem in* $\mathbf{n}_S = [\mathbf{n}_\sigma : \sigma \in S]$, *for* $\mathbf{I}_S = [\mathbf{I}_\sigma : \sigma \in S]$ *:*

Find $\mathbf{n}_S^*$ *such that,* $\forall \mathbf{n}_S$ *that are admissible,* $\mathbf{I}_S(\mathbf{n}_S^*) .(\mathbf{n}_S - \mathbf{n}_S^*) \geq 0$. (4.3)

If the functions O_r and H_σ are continuous, $\mathbf{I}_S$ is too, which guarantees the existence of a solution if the problem is feasible (in particular, total supply must not be lower than total demand). This demonstrates the existence of a residential equilibrium.

Extremal formulation and property of uniqueness

The above treatment has been adapted from the model for traffic assignment on a transport network developed by Leurent (1993, 1996). In the case where there is only one demand segment σ, which makes it possible to represent a monocentric model with a distribution of incomes but not household sizes, the function $\mathbf{I}_S$ is derived from a potential function, namely:

$$J_\sigma = \sum_{r=1}^{R_\sigma} \int^{n_{\sigma r}} a_{\sigma r}(\theta)\,d\theta + Q_\sigma \sum_{r=1}^{R_\sigma} b_{\sigma r}[\eta_\sigma(\tfrac{N_{\sigma r}}{Q_\sigma}) - \eta_\sigma(\tfrac{N_{\sigma r-1}}{Q_\sigma})],$$

where $\eta_\sigma(x) = \int^x H_\sigma^{-1}(\theta)\,d\theta$. (4.4)

In fact, the function J_σ has as the following partial derivative:

$$\frac{\partial J_\sigma}{\partial n_{\sigma r}} = a_{\sigma r}(S_r) + \sum_{\ell=1}^{R_\sigma} b_{\sigma \ell}[H_\sigma^{-1}(\tfrac{N_{\sigma \ell}}{Q_\sigma}) 1_{\{R_\sigma > \ell \geq r\}} - H_\sigma^{-1}(\tfrac{N_{\sigma \ell -1}}{Q_\sigma}) 1_{\{\ell > r\}}]$$

$$= a_{\sigma r} + \sum_{\ell=r}^{R_\sigma -1} b_{\sigma \ell}\, H_\sigma^{-1}(\tfrac{N_{\sigma \ell}}{Q_\sigma}) - \sum_{\ell=r+1}^{R_\sigma} b_{\sigma \ell}\, H_\sigma^{-1}(\tfrac{N_{\sigma \ell -1}}{Q_\sigma})$$

$$= a_{\sigma r} + \sum_{\ell=r}^{R_\sigma -1} (b_{\sigma \ell} - b_{\sigma \ell +1})\, H_\sigma^{-1}(\tfrac{N_{\sigma \ell}}{Q_\sigma})$$

Furthermore $\dfrac{\partial^2 J_\sigma}{\partial n_{\sigma s} \partial n_{\sigma r}} = 1_{\{r=s\}} \dfrac{\partial a_{\sigma r}}{\partial n_{\sigma r}} + Q_\sigma^{-1}.\sum_{\ell=1}^{R_\sigma -1} (b_{\sigma \ell} - b_{\sigma \ell +1}) 1_{\{\ell \geq r\}} 1_{\{\ell \geq s\}}\, \dot{H}_\sigma^{-1}(\tfrac{N_{\sigma \ell}}{Q_\sigma})$,

so $\sum_{r,s} \dfrac{\partial^2 J_\sigma}{\partial n_{\sigma s} \partial n_{\sigma r}} x_s x_r = \sum_r \dfrac{\partial a_{\sigma r}}{\partial n_{\sigma r}} x_r^2 + Q_\sigma^{-1}.\sum_{\ell=1}^{R_\sigma -1} (b_{\sigma \ell} - b_{\sigma \ell +1})\, \dot{H}_\sigma^{-1}(\tfrac{N_{\sigma \ell}}{Q_\sigma}) X_\ell^2$ for $X_\ell = \sum_{r=1}^{\ell} x_r$.

All the terms in the sum are below zero because $\Delta b \leq 0$, H_σ is an increasing function, so its reciprocal is increasing too and it has a non-negative derivative $\dot{H}_\sigma^{-1}$, while $\partial a_{\sigma r} / \partial n_{\sigma r} = -s_r \dot{O}_r^{-1}$ which is non positive since the inverse supply function is increasing hence its derivative is non negative. J_σ is therefore concave. This entails the uniqueness of the residential equilibrium (with regard to the components whose diagonal coefficient in the Hessian matrix $[\partial^2 J_\sigma / \partial n_{\sigma s} \partial n_{\sigma r} : r, s \in R_\sigma]$ is strictly negative).

In the case where there are several demand segments, the coefficients $[b_{\sigma r} : \sigma \in S]$ for a given option r are heterogenous, so the term $a_{\sigma r}$ in the attractiveness function $I_{\sigma r}$ can no longer be included in an objective function. The reason for this is that $\partial a_{\sigma r} / \partial n_{\sigma' r} = -b_{\sigma r} s_r \dot{O}_r^{-1}$ so if $b_{\sigma r} \neq b_{\sigma' r}$ it follows that $\partial a_{\sigma r} / \partial n_{\sigma' r} \neq \partial a_{\sigma' r} / \partial n_{\sigma r}$.

The heterogeneity of the demand segments in the case of residential assignment is analogous to that of different classes of traffic in the case of assignment on a transport network (in particular passenger cars and trucks), in which case for each class and network arc there is a journey time that depends on the volumes of the different classes of vehicles, converted into passenger car units by means of specific coefficients of equivalence. From our practical experience of these models we know that the heterogeneity within each segment (cf. income distribution) makes it possible to compute a satisfactory equilibrium (Leurent, 1995b), while a variety of solutions is obtained with multiclass assignment models with homogeneous segments (Wynter, 1995).

Resolution algorithm

Residential equilibrium can be found by an algorithm that is taken from the assignment of traffic on a network. A natural first approach is to apply the "historical" algorithm developed by Beckmann *et al* (1956), i.e. the Method of Successive Averages (MSA) as it has been adapted for the dual criteria assignment model (Leurent, 1995a).
The MSA algorithm performs successive iterations, which are given the index k. The price of the options is updated during each iteration on the basis of the clienteles $[S_r^{(k)} : r \in R]$. Each segment $\sigma \in \mathrm{S}$ is then processed as follows:
determination of the coefficients $b_{\sigma r}$ and $a_{\sigma r}$ $\forall r \in R_\sigma$.
determination of the efficiency intervals $[\underline{\rho}_{\sigma r}^{(k)}, \overline{\rho}_{\sigma r}^{(k)}]$.
from which an auxiliary state for the clientele is derived: $n_{\sigma r}'^{(k)} = Q_\sigma[\mathrm{H}_\sigma(\overline{\rho}_{\sigma r}^{(k)}) - \mathrm{H}_\sigma(\underline{\rho}_{\sigma r}^{(k)})]$.
Once all the segments have been processed, the next state is determined by a convex combination of the present state of order k and the auxiliary state, whose combination coefficient $\zeta_k \in]0,1[$ is predetermined:
$\forall \sigma \in \mathrm{S}, \forall r \in R_\sigma,\ \ n_{\sigma r}^{(k+1)} = (1-\zeta_k) n_{\sigma r}^{(k)} + \zeta_k n_{\sigma r}'^{(k)}$,
$\forall r \in R,\ \ S_r^{(k+1)} = \sum_{\sigma : R_\sigma \ni r} n_{\sigma r}^{(k+1)} = (1-\zeta_k) S_{\sigma r}^{(k)} + \zeta_k S_{\sigma r}'^{(k)}$.
Convergence towards equilibrium is assessed on the basis of the duality gap of the variational inequality, $\mathrm{DG}^{(k)} = \sum_{\sigma \in \mathrm{S}} \sum_{r \in R_\sigma} \mathrm{I}_{\sigma r}^{(k)} . (n_{\sigma r}'^{(k)} - n_{\sigma r}^{(k)})$. This criterion is non-negative by construction, is only cancelled at one equilibrium, and by continuity a very small value indicates that an equilibrium state has almost been reached. If the criterion is sufficiently small, the algorithm is stopped, otherwise another iteration is performed.
For the initial state at $k = 0$, to generate values of $S_r^{(0)}$ therefore the prices $p_r^{(0)}$ and attractiveness values $\mathrm{I}_{\sigma r}^{(1)}$, it is possible to fix the values of $n_{\sigma r}^{(0)}$ for instance as a uniform distribution, $n_{\sigma r}^{(0)} = Q_\sigma / \mathrm{Card}(R_\sigma)$.
The sequence (ζ_k) must decrease towards zero, but not too rapidly in order to redistribute demanders efficiently between the options at each iteration.

APPLICATION TO THE PARIS REGION

We now present an application of our model to the Paris region.

Application settings

Our study area is the "région Île-de-France", the administrative region where Paris is located. We will also refer to the Île-de-France as the Paris region for the sake of clarity. The Île-de-France covers 12 million km². In 1999, its population amounted to 10 million people, which represented at the time approximately 15% of the French population.

Zone system

The Paris region is divided in 36 geographical zones according to the zone system developed by the DRIEA (La Corte, 2006). This system is based on the aggregation of smaller transportation analysis zones. The 36 zones are relatively homogenous in terms of population. This means that the size of each zone decreases with density, hence smaller zones in Paris and larger zones in the outer ring.

Household segments

The scope of this first application is limited to "working households", that is households whose household head is employed. In 1999 (which is our reference year for the application) working households accounted for around 80% of the overall regional population according to Census data.
Working households are stratified by size according to the following categories:

Table 2 – Household segments in the metropolitan area

	1 person	2 persons	3 persons	4 persons	5 persons and more
Number of households	773 760	852 525	651 109	609 409	352 173

Source : 1999 Census

Households are also regrouped according to the household head's employment zone. All in all, we have 5*36=180 household classes. While income distribution is assumed to be lognormal in all cases, the mean and standard deviation can vary for each class and were estimated using the Enquête Globale de Transport (EGT) 2001-2002 (the main transportation survey for the Paris region).

Figure 1 shows for each zone the number of jobs (in circles) and the ratio between the number of jobs and the housing stock. The zones of central Paris and the zones containing Roissy and the scientific cluster of Saclay have a relatively high quantity of jobs. In comparison, peripheral zones have a rather residential function.

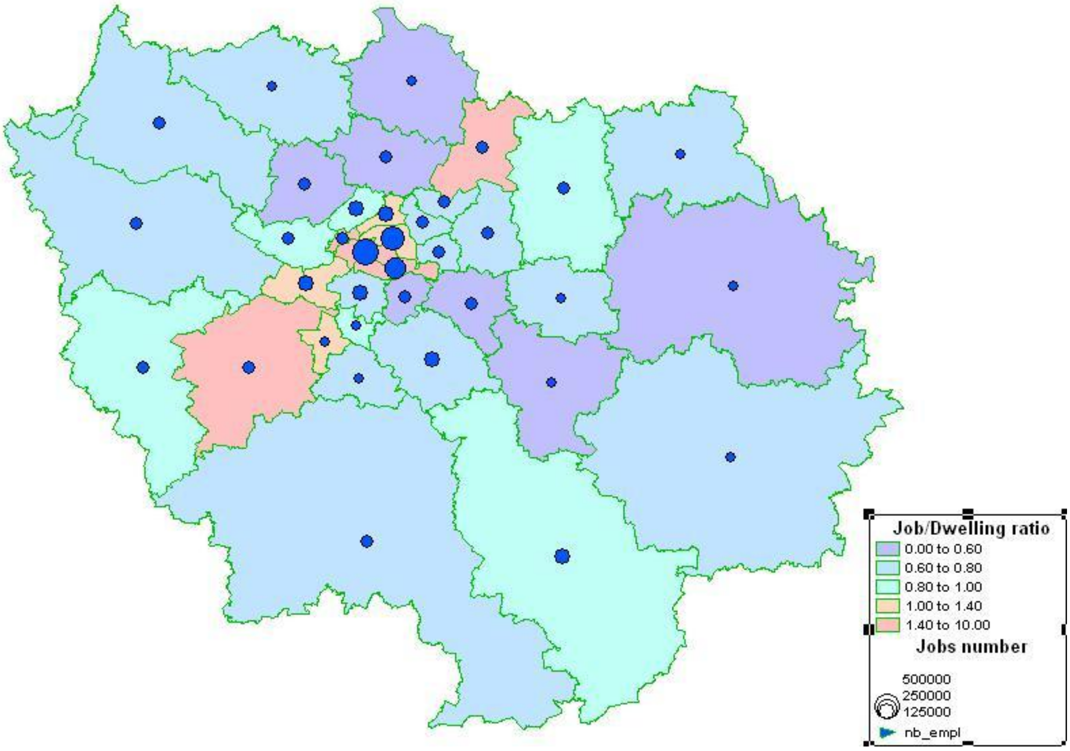

Fig 1: Job - dwelling ratio and number of jobs per zone, 1999

Dwelling segments

In accordance with the segmentation of demand, we stratify dwellings according to the number of rooms. All dwellings are assumed to have the same size within each segment, which is the observed mean for the corresponding category.

Table 1 : Number of homes occupied by working households and average dwelling size for the Paris region, 1999

	1 room	2 rooms	3 rooms	4 rooms	5 rooms and +
Number of dwelling units	332 460	613 460	810 980	718 240	687 520
Average floor area	29.7	46.75	68.7	86.5	140

Source : 1999, Census

Transportation costs

We only consider road transportation in this first application. For each O-D pair, monetary costs and travel times are computed for the morning rush hour using a road assignment model. The model is implemented in TransCAD using 2004 data from the DRIEA.

Parameter settings

Housing: the starting rent is set at 0.25€/m²/day. User cost is set at 3€/m²/day per dwelling. Regarding housing supply, the price elasticity is set to 0.1 for inner Paris, 0.2 for the inner ring and 0.3 for the outer ring. Compared to observed stocks (occupied by working households), initial volumes are cut off by 20% for inner Paris, kept constant for the inner ring and raised by 20% for the outer ring.
Income distribution: for all household classes, the minimum income for the log-normal distribution is set to 10€/working day in accordance to observed data in the EGT (as a way of comparison, the official minimum wage was actually 48.5€/day at the time).
Utility function: we set the parameters of the Stone-Geary utility function $U \propto (s - vs_0)^{\alpha} q^{\beta} z^{\gamma} \tau^{\delta}$ as follows: α =0.12, β= 0, γ=0.5, δ=0.2. Dwelling quality (q^{β}) is not considered in this application. Lastly, the minimal floor space is set to 10m² per person. All parameters were were estimated using the 2002 French housing survey (Coulombel, 2012).

Computation results

The model was coded in C++. For our case study, around 10 000 iterations are necessary to reach convergence. On a 2.77GHz processor, this involves a computational time of around 5 minutes. At convergence, the value of the duality gap is about 5000 and the value of the attractiveness function factor is approximately 100. Using the variational inequality, this means that marginal variations add up to about 50 units per iteration, to be compared to the total stock of around 3 million units.

The combination step used in the algorithm is $\xi_k = \dfrac{1}{\frac{k}{2}+1}$, where k is the current iteration.

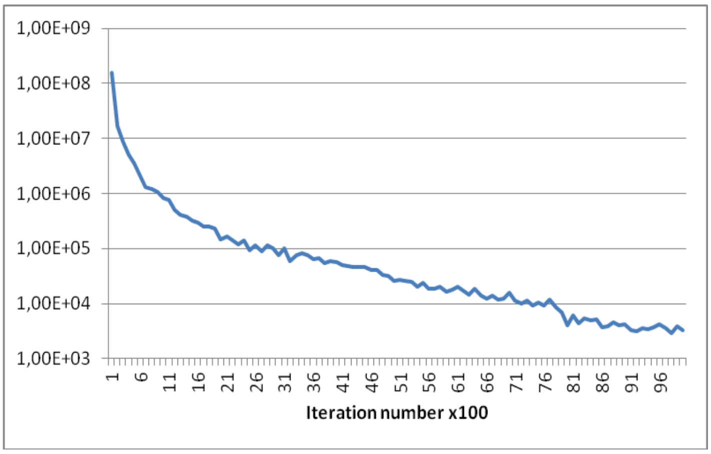

Figure 2 – Variation of the duality gap with the number of iterations

Preliminary results of the calibration prove promising: the model manages to correctly fit observed data regarding zonal population (Figure 3). The main gaps are observed for Paris (zones n°29, 30, 31) where population is underpredicted by the model, and for the zones of Roissy and Saclay (n°11, 21). The high number of jobs in these zones leads to an overestimation of population compared to what is observed.

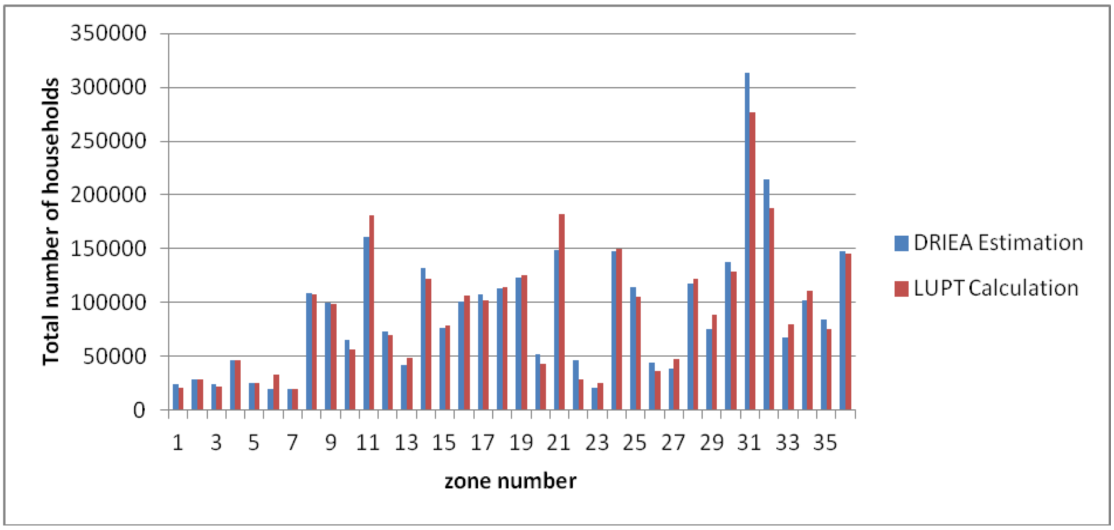

Figure 3 – Comparison between model results and observed data

Figure 4 presents spatial variations of prices for 3-room homes. Housing prices increase with job accessibility. A brief comparison with observed home sales prices further validates our model. According to the notaries' database for the Paris region (called BIEN), in 2001 the ratio of home prices was 0.73 between the inner ring and Paris and 0.6 between the outer ring and Paris. Our model predicts ratios of 0.8 and 0.58 respectively. In addition, at equilibrium our model predicts an average rent of 9.6€/m²/month in inner Paris, against 12€/m²/month according to official data from the INSEE.
Relatively to 3-room homes, the prices of the other dwellings are as follows: 1 room +22%, 2 rooms +1%, 4 rooms +5% and 5 rooms and more +10%. 1 room homes are significantly more expensive as 1 and 2 person households compete over a relatively scarce supply.

One especially promising feature is that the model also manages to reproduce the decrease of household size with distance to Paris (Figure 5). Small households live in central Paris while large households prefer to settle in the suburbs, where housing is more affordable. Still, Figure 6 shows that many small households (2, 3 persons) live in suburbs with a comfortable floor area.
Last but not least, the model correctly reproduces the high commuting times of suburban residents, while households living in Paris enjoy lower commuting time thanks to the high job accessibility of the capital (Figure 7).

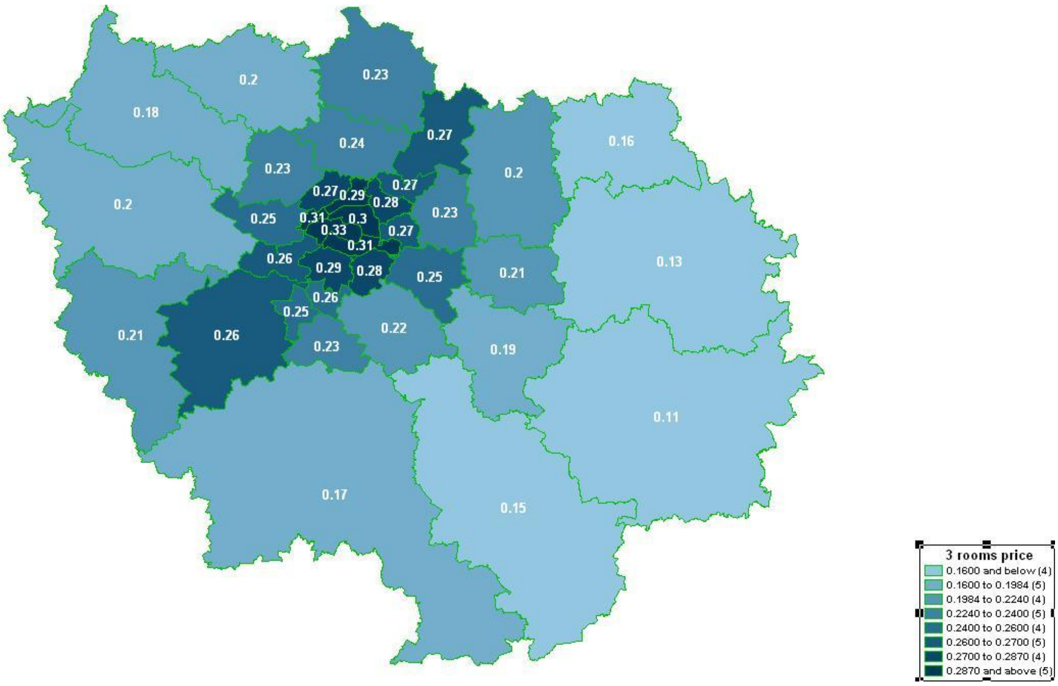

Figure 4 – Rental prices for 3-room homes

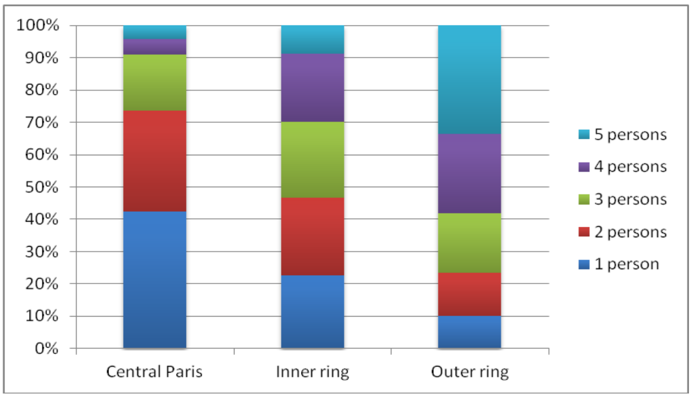

Figure 5 – Household size distribution

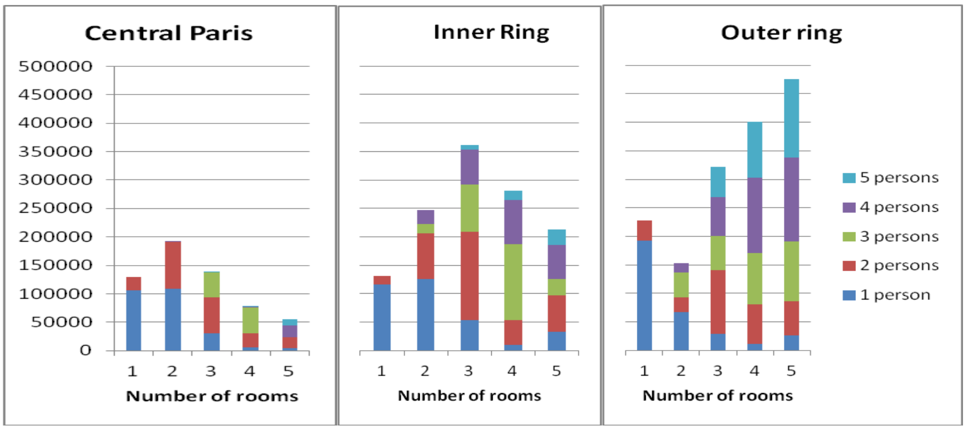

Figure 6 –Household size distribution per dwelling type

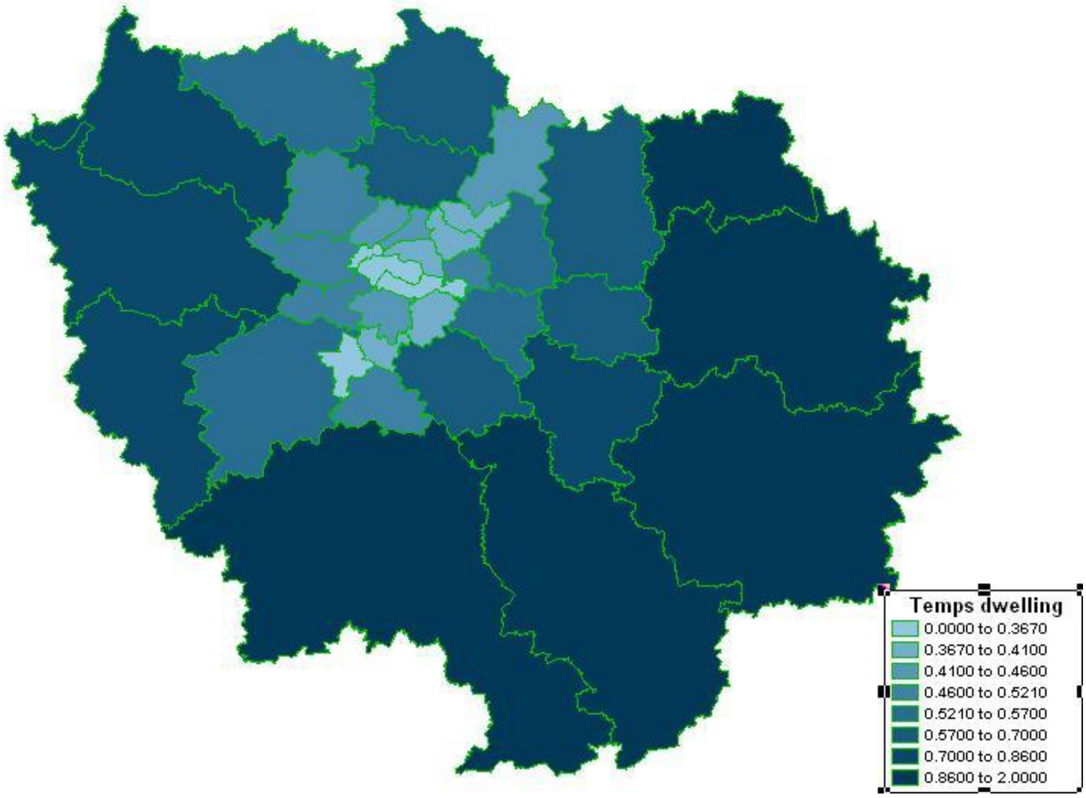

Figure 7 – Average commuting time (in hours)

A test scenario of an increase in fuel prices

We test the model behavior and sensitivity with a simple scenario of an increase in fuel prices. Fuel prices are set to 2 €/litre, against 1€ in 1999.
Table 3 summarizes the mean relative change in commuting time by household segment. Overall, the commuting time decreases (by almost 2%), which is conform to intuition. Interestingly, the impact of the increase in fuel prices differs according to the size of the household. Large households have a larger basic need for housing than small households (the minimum is 10 m² per person), meaning that they are on average less rich than small households. Large households are thus forced to move closer to their jobs, be it at the cost of slightly reducing their home size. On the contrary, smaller households are relatively less impacted by higher fuel prices, and can occupy the new vacant dwellings. This reinforces the attractiveness of the medium dwellings in the employment zones, leading to higher prices for this category, contrary to peripheral zones dwellings (Figure 8). More specifically, small dwelling rent increases more than 25% in the central metropolitan area, while large dwelling price increases only up to 5%. On the contrary, the decrease in the peripheral zones is similar for all dwelling types.

Differences in housing stock between the reference situation and the higher fuel price scenario confirms the first explanations. Households leave the outer ring as an attempt to get closer to their job location. Some zones lose more than 5% of their population. Indeed, they would prefer to live in central Paris but the low housing supply elasticity limits the capacity of the central zones to accommodate for more population. Thus, households move to the inner ring,

near the central zones. The larger the household, the more the volume difference in the centre increases.
To conclude, through the presented indicators, not only do we have a validation of the correct behavior of the model, but we also gain more understanding in the disaggregated behavior of households.

Tab 3 - Transportation time per household segment

	1 person	2 persons	3 persons	4 persons	5 persons	Total
Transportation Time difference	+0.64%	+0.45%	-0.92%	-4.3%	-6%	-1.9%

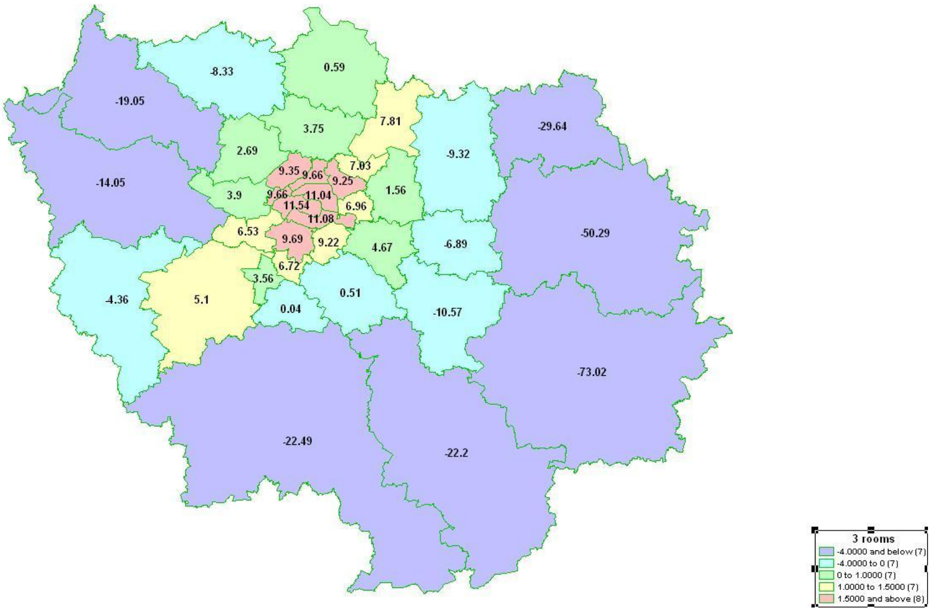

Figure 8 – Relative change in rental prices, 3-room homes

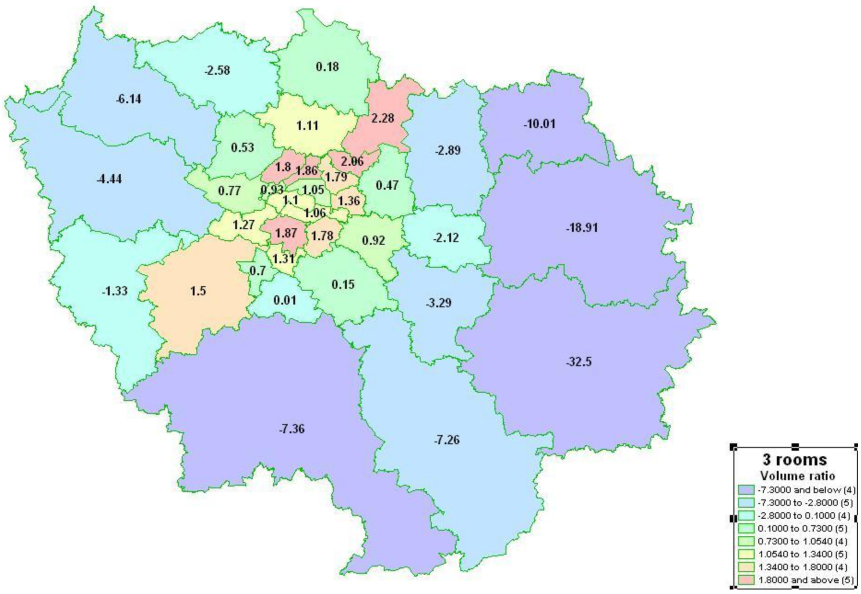

Figure 9 – Relative change in housing stock, 3-room homes

CONCLUSION

We have modelled the structure of housing supply according to type and zone, and the structure of demand according to household size, job location and income. We have modelled the microeconomic behaviours of demanders and macroeconomic behaviours on the supply side. The structural features that are represented are fundamental for the medium-term operation of an urban housing market: they provide a minimum core which must feature in any operational (or pre-decisional) application.

Our model deals with the disaggregation of demand. The inherent complexity has been reduced mathematically and algorithmically owing to a specific aggregated treatment.

The model could be extended in many ways – economically, mathematically or algorithmically. We shall simply mention, from the economic standpoint, the representation of households with no working member or several workers, the inclusion of social housing, the distinction between the rented sector and the "owner-occupier" sector in the case of the private sector property market, and the inclusion of complex supply behaviours, etc.

As it stands, the model provides a good compromise between explanatory power and processing simplicity. A first application has been implemented on a real case study with several household and dwelling segments. The ensuing results confirm the relatively accurate behaviour of the model and its correct estimations despite the limited number of parameters in the utility function of the model. Withal, future improvements will bear on integrating both dwelling and neighborhood quality in the model, thus leading to probably more complex utility functions.

REFERENCES

Anas A. and Liu Y. (2007), A regional economy, land use, and transportation model (RELU-TRAN): Formulation, algorithm design, and testing, Journal of Regional Science, Vol.47 (3), pp.415-455

Beckmann M, McGuire CB, Winsten CB (1956). Studies in the economics of transportation. Report to the Coles Commission. Yale University Press.

Delons J, Coulombel N and Leurent F (2009). Pirandello: an integrated transport and land-use model for the Paris area. Working document submitted to TRB Congress 2010.

David Simmonds Consultancy in collaboration with Marcial Echenique and Partners Limited (1999), Review of Land - Use/Transport Interaction Models, Reports to The Standing Advisory Committee on Trunk Road Assessment (SACTRA)

King D. (2011), Developing densely: estimating the effect of subway growth on New York City land uses, The Journal of Transport and Land Use, Vol.4 (2), pp.19-32

Leurent, F. (1993). Cost versus Time Equilibrium over a Network. *Eur. J. of Oper. Res.,* Vol. 71, pp. 205-221.

Leurent, F.M. (1995a). Un algorithme pour résoudre plusieurs modèles d'affectation du trafic : la méthode d'égalisation par transvasement. *Les Cahiers Scientifiques du Transport*, Vol. 30, pp. 31-49.

Leurent, F (1995b). *Comparaison de deux modèles d'affectation du trafic*. Rapport sur convention Sétra. INRETS, Arcueil.

Leurent, F.M. (1996). The Theory and Practice of a Dual Criteria Assignment Model with a Continuously Distributed Value-of-Time, in *Transportation and Traffic Theory*, Lesort J.B. (ed), pp. 455-477. Pergamon, Exeter, England.

Leurent F (2012) Les modèles d'usage du sol et transport : où la géographie et l'économie se rejoignent. In Hégron G (ed), La modélisation de la ville : du modèle au projet urbain. Numéro special de la Revue du CGDD, Avril, pp. 154-171.

Lowry IS (1964) A Model of Metropolis. RAND Memorandum 4025-RC.

Levinson D. (2007), Density and dispersion: the co-development of land use and rail in London, Working paper

Salvini P. and Miller E.J. (2005), ILUTE: An Operational Prototype of Comprehensive Microsimulation Model of Urban Systems, Network and Spatial Economics, Vol.5, pp.217-234.

Simmonds D., Waddell P. and Wegener M. (2011), Equilibrium v. dynamics in urban modelling, Paper presented at the Symposium on Applied Urban Modelling (AUM 2011), University of Cambridge, 23-24 May 2011

Waddell P. et alii (2003), Microsimulation of Urban Development and Location Choices: Design and Implementation of UrbanSim, Networks and Spatial Economics, Vol. 3 (1), pp. 43-67.

Wynter L. (1995), Contributions à la théorie et à l'application de l'affectation multi-classe du trafic, Thèse de doctorat de l'ENPC, novembre.