An equilibrium mode-split model of work trips along a transportation corridor

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1. INTRODUCTION

Within the sphere of research associated with transportation planning, the problem of equilibration - one of solving a set of demand and supply (level-of-service) equations - has often been left untouched. This was so primarily because of the conceptual complexity of the problem and secondarily because of the high computational cost of implementing an equilibrium model. Thus, in many applications of transportation demand studies, some kind of supply inelasticity (to travel volumes) assumptions were made. And in those applications where congestion effects were felt to be significant enough to be decisive, the Incremental Assignment technique was invariably used. This technique is computationally quite inexpensive, but is ad hoc and has problems with convergence.

It is in the last decade or so that we begin to see an increased vigor in the development of equilibrium models. The seminal paper of Dafermos and Sparrow [1] can be identified as a turning point and the start of more mathematical analyses of the equilibration problem. For a unified approach to equilibration methodologies, viewed as solutions to an optimization problem, see, for example, the paper by Nguyen [5].

The new generation of iterative equilibrium models are, unfortunately, still quite expensive to use in modeling large transportation systems, and because of their highly aggregate nature, are unsatisfactory from the point of view of behavioral theories of travel demand such as that developed by McFadden [3], [4]. The object of this paper is to suggest a way in which an equilibrium model can be developed, whose components are a set of supply equations and a set of *disaggregate* mode-choice equations. We restrict our attention to the analysis of work-trip mode splits over a *transportation corridor.*

In view of the remarkable success of the applications of Scarf algorithm [6] in computing economic equilibria (see, for example, Shoven and Whalley [8], [9]), we have cast the equilibration problem not as an optimization problem, but as a problem of computing the fixed point of some appropriate mapping.

Section 2 details the components of the model. Section 3 summarizes the key concepts involved in the Scarf algorithm, and its application to the model developed in section 2. In section 4 we indicate briefly our limited experience in applying the model to an actual transportation planning problem.

2. A REFORMULATION OF THE EQUILIBRATION PROBLEM

2.1 Segmentation of the Corridor

Let us begin by reiterating the problem at hand. Given the information on home and work locations; some socioeconomic characteristics which are assumed to completely describe the utility-maximizing workers; the distribution of tastes among these workers; the characteristics of the transportation corridor along which home and work locations are scattered, predict the equilibrium work-trip flow pattern[1] along the corridor. Such a result, being an indispensible prerequisite for any cost-benefit analysis, hardly requires further elaboration as to its importance.

To reduce the number of "markets" where supply and demand have to be equilibrated, we divide the corridor into large segments, each of which consists of several traffic zones. The segment boundaries should be chosen along the most natural geographic lines perpendicular to the "axis" of the corridor (for example, highway intersections might be suitable points through which such boundaries pass). Figure 1 gives a schematic representation of the segmentation of the I-580 corridor in the San Francisco Bay Area.

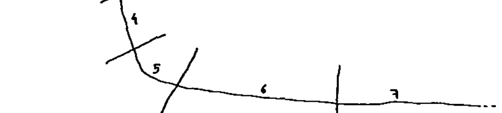

Figure 1 - Schematic Segmentation of the I-580 Corridor in the San Francisco Bay Area

2.2 Travel Demand

Let us assume that each of the workers in our system chooses his mode of transportation to and from work so as to maximize his utility (minimize his discomfort). The individual has a utility function that can be written in the form

$$u(y,t) = v(y,t) + \varepsilon(y,t) \qquad (1)$$

where we have assumed that utility depends only on travel time t and an index of socioeconomic characteristics y. This assumption is artificial and is made only for the sake of ease of exposition.

Adding monetary cost of travel and other variables into equation (1) is straight forward, and will have the sole effect of clouding up the structure of the problem. v(y,t) can be interpreted as the "representative" taste of the population, and $\varepsilon(y,t)$ is a stochastic term representing taste variation among individuals. As is usually done in empirical work, we hypothesize a linear dependence of v on y and t:

$$v(y,t) = ay + bt \tag{2}$$

a and b being coefficients which can be, and have been, we assume in this paper, statistically estimated. Under the assumption that the values $\varepsilon(y,t)$ are independently identically distributed with a Weibull distribution, McFadden [3] has shown that the probability of individual i choosing mode m from a common alternative choice set M is given by the multinomial logit model

$$P_m^i = \frac{\exp(bt_m^i + ay_m^i)}{\sum_{l \in M} \exp(bt_l^i + ay_l^i)} \qquad \begin{matrix} i=1,\ldots,I \\ m=1,\ldots,M \end{matrix} \tag{3}$$

where I is the total number of individuals in the system, and M denotes both the total number of modes available to any individual, and the alternative choice set which is common to all individuals in the system.

2.3 Supply Equations

Following the convention in transportation literature, we distinguish between two types of supply relations or relations between travel times and volumes over streets and highways: main-mode supply relations and access/egress supply relations. We derive main-mode supply relations for each mode over each *segment* in the corridor, by either extending the single bottleneck formulation of May and Keller [2] and Small [10] to a collection of roads or by an aggregation of individual road performance characteristics, a technique which utilizes Wardrop's [12] first principle and amounts to horizontal (vertical) summation of parallel (consecutive) links' "travel time vs. volume" curves.

To use the technique of May-Keller-Small, we need to identify the "bottleneck" for the segment as a whole and for each main modes of travel. We thus derive a restraining capacity for each segment and main mode which "meters" the traffic on to the segment. Under the customary assumption that the peak period travel volume is distributed uniformly over the peak period of duration P hours, it has been shown that the *average* travel time (x_{jm}) for mode m over segment j is given by:

$$x_{jm} = \max[0, \frac{B_{jm}}{C_{jm}} - 1]\ \frac{P}{2} + T_{jm}$$

where $$B_{jm} = \sum_{n=1}^{M} \gamma_n^m D_{jn} \quad .$$

B_{jm} is a weighted sum of travel demands D_{jn} of all the M modes in segment j, the weights γ_n^m being the "equivalence" factors of the modes; C_{jm} is restraining capacity of segment j; and T_{jm} is the "free-speed" travel time of mode m in segment j.

As an example, consider the following model with three modes:

m = 1: Auto
m = 2: Express Bus
m = 3: Local Bus

Assume that the modes Auto and Express Bus use freeways and the mode Local Bus uses arterial roads. Then, the travel time for mode 1 over freeway segment j is given by:

$$x_{j1} = \max[0, \frac{B_{j1}}{C_{j1}} - 1]\ \frac{P}{2} + T_{j1}$$

where

$$B_{j1} = \gamma_1^1 D_{j1} + \gamma_2^1 D_{j2} + \gamma_3^1 D_{j3}$$

and

$\gamma_1^1 = 1$

γ_2^1 = the number of car-equivalents of an Express Bus

$\gamma_3^1 = 0$

γ_3^1 is zero since Local Bus does not use freeways and hence does not affect the travel time on Auto. It should be noted that since Express Bus also uses freeways, the travel time for Express Bus, x_{j2}, is given identically by x_{j1}: [3] $x_{j2} \equiv x_{j1}$.

The second alternative, the aggregation of individual road link supply relations of the segment into a segment relation, is a generalization of the single bottleneck concept of a segment. More realistically, there are more than one bottleneck per segment. Each "road", in fact, has a capacity beyond which travel time shoots up very fast. Thus, we have, for each segment of the corridor, a structure consisting of roads with different supply relations. For example, a segment might have the following structure of roads.

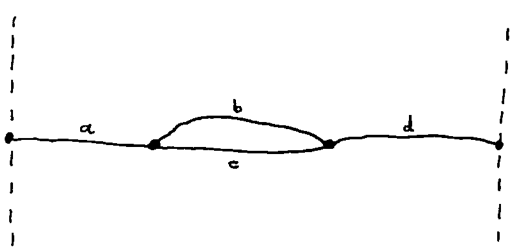

Figure 2. A Segment Made Up of Four Roads a, b, c, d Each Characterized by its Own Time-Volume Curve x_a, x_b, x_c, x_d

Under the assumption that conditional on any chosen mode, a worker will pick the shortest path available, it is easy to see that Wardrop's first principle holds. Thus, two parallel roads characterized by two different supply curves x_a, x_b is equivalent to one road characterized by a supply curve x_c which is a horizontal sum of the supply curves x_a and x_b. Similarly, two roads in sequence is equivalent to a single road whose supply curve is a vertical sum of the two original curves. Figure 3 illustrates these cases.

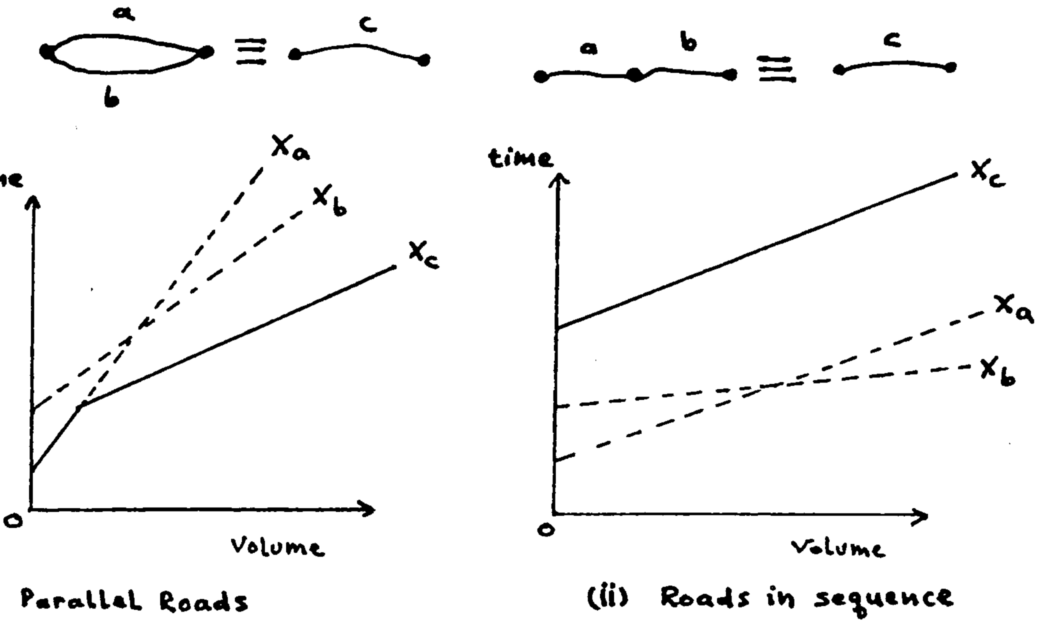

Figure 3 - Note that x characterizes road i

We can apply such simple schemes sequentially and reduce even very complex structures into a single "road" characterized by a single supply relation. In this way, we can derive *segment* supply relations for all segments and modes (the same kind of arguments can be used to construct segment relations for guideway modes). For example, denoting the operation in Figure 3 (i) by $x_a \textcircled{P} x_b = x_c$ and that in Figure 3 (ii) by $x_a \textcircled{S} x_b = x_c$, we derive an expression for the segment supply curve for the segment shown in Figure 2, in terms of its component road supply curves x_a, x_b, x_c, x_d, as follows:

$$x = x_a \textcircled{S} (x_b \textcircled{P} x_c) \textcircled{S} x_d$$

To contrast the two formulations of segment supply functions, consider a segment that consists of two roads in sequence with supply curves x_a and x_b. Now the single-bottleneck approach identifies the restraining capacity which, in this case, happens to be in road b, and assumes that there is no capacity restraint in a. Hence, a is characterized by a constant "freespeed" travel time, T_a, and the segment supply curve is derived by a vertical addition of T_a and x_b. We compare this to the result of the Wardrop scheme of vertically adding x_a and x_b, in Figure 4.

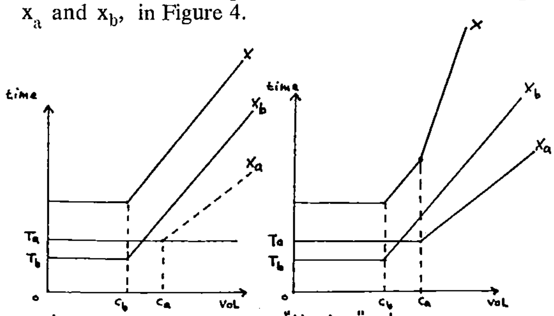

Figure 4

We thus see that in the range of travel volumes from zero to C_a, the single "bottleneck" and "multiple bottleneck" formulations give identical travel times. Beyond C_a the travel times are different, the multiple bottleneck travel time being larger than the single bottleneck travel time. Which one is correct? The answer is not clear. If the two consecutive road links are independent, as in the case of a low volume arterial street governed with unsynchronized traffic signals, then the multiple bottleneck formulation (every signal) is approximately correct. However, if the consecutive links are not independent, as in the case of freeways, then the single bottleneck version is approximately correct. In real world situations both type of cases occur intertwined and it is an empirical question which of the two schemes is better. The single bottleneck formulation is very attractive because of its simplicity and small data requirements. Its success will depend in a large measure on whether the delays in a segment of a corridor are due to congestion or traffic control devices (traffic lights, stop signs, etc.) normally found on arterial streets. Small [10] has used the point bottleneck model to a freeway segment several miles long with apparent success. The prevalence of signals and stop signs and other disturbances on arterials suggest that perhaps a "marriage" between the two methods is the best solution.[4]

An individual's path from home to work (and vice versa) is usually broken down into an access component, a main-mode component and an egress component. We assume that the access[5] part of the trip on any mode will always be from home to the *nearest* "main-mode entrance" (highway entrance, bus station, and so forth) plus the trip from the main-mode exit *nearest* to the work location itself (the trip home from work is dichotomized into access and main-mode in a similar fashion). This assumption eliminates the need to explicitly model each worker's choice probabilities of main-mode entrances available to him. Checks made on a subset of the Urban Travel Demand Forecasting Project's sample survey of about 800 workers in the San Francisco Bay Area in 1975 indicate that the assumption holds up quite well; around 85% of BART riders chose the stations closest to their homes.

2.4 The Complete Model

We are now almost ready to put together our demand and supply formulations into a complete model. However, note a crucial problem in equilibration: while demand is in terms of individual work-home trip variables, our supply formulation is in terms of segment variables. One way to overcome this inconsistency is to make the following assumption. We assume that the congestion effects due to a vehicle[6] entering a segment of length L_j a distance l_{ij} from the boundary towards which it is going, is equivalent to those due to a fraction δ_{ij} of a vehicle traversing the segment completely from boundary to boundary.

$$\delta_{ij} = \frac{l_{ij}}{L_j} \tag{4}$$

What we are assuming basically is that a lot of congestion over a part of the segment is equivalent to a milder congestion over the whole of the segment. Our assumption enables us to aggregate individual demands for the various modes into segment demands for all modes.

$$D_{jm} = \sum_{i=1}^{I} \delta_{ij} P_m^i \tag{5}$$

Note further that

$$t_m^i = \sum_{j=1}^{J} \delta_{ij} x_{jm} + \bar{x}_m^i \tag{6}$$

where x_{jm} is the travel time over segment j by mode m; $\bar{x}_m^i$ is the access travel time, which depends only on the individual's characteristics (i.e., where his home and work locations are) and the main mode "entrances" for mode m; and J is the total number of segments.

Our model is now completely specified. It is described by the following equations.

$$P_m^i = \frac{\exp[b(\sum_{j=1}^{J} \delta_{ij} x_{jm} + \bar{x}_m^i) + ay_m^i]}{\sum_{l \in M} \exp[b(\sum_{j=1}^{J} \delta_{ij} x_{jl} + \bar{x}_l^i) + ay_l^i]} \tag{7}$$

$$D_{jm} = \sum_{i=1}^{I} \delta_{ij} P_m^i \tag{5}$$

$$x_{jm} = S_{jm}(D_{j1}, \ldots, D_{jM}; T) \tag{8}$$

$$\bar{x}_m^i = \bar{S}_m(w^i; D_{j_i 1}, \ldots, D_{j_i M}; T) \tag{9}$$

Equations (7), (5), (8), (9) hold for $i=1,\ldots,I$; $j=1,\ldots,J$; $m=1,\ldots,M$

T is a vector characterizing the transportation system characteristics; j_i denotes the segment where access occurs; and w^i is a variable characterizing the work and home locations of individual i. Equation (8) is a representation of the result of the segment supply derivation in our previous section. Equation (9) can be derived, for example, in a way that parallels the approach of Talvitie-Dehghani [11].

Typically, however, there are hundreds of thousands of workers using the corridor so that a straight individual enumeration, as implied by the above model, becomes too cumbersome to perform. We are forced, therefore, to use only a sample of the whole population. One easy scheme is the following. Sample individuals at a rate Θ from the given trip table. Observe the sampled individual's home and work locations, his socioeconomic characteristics, and the nearest main-mode entrances and exits. From this information, we obtain:

$$\delta_{kj}; \bar{S}_m(w_j^k; \ldots; T); y_m^k \qquad k=1,\ldots,K \quad j=1,\ldots,K \quad m=1,\ldots,M$$

where K is the total number of sampled individuals, each of whom is identified with the index k.

Our model is then described by the following equations.

$$D_{jm} = \frac{1}{\Theta} \sum_{k=1}^{K} \delta_{kj} P_m^k \tag{10}$$

$$P_m^k = \frac{\exp[b(\sum_{j=1}^{J} \delta_{kj} x_{jm} + \bar{x}_m^k) + ay_m^k]}{\sum_{l \in M} \exp[b(\sum_{j=1}^{J} \delta_{kj} x_{jl} + \bar{x}_l^k) + ay_l^k]} \qquad k=1,\ldots,K \quad m=1,\ldots,M \tag{11}$$

$$x_{jm} = S_{jm}(D_{j1}, \ldots, D_{jM}; T) \tag{8}$$

$$\bar{x}_m^k = \bar{S}_m(w^k; D_{j_k 1}, \ldots, D_{j_k M}; T) \tag{9}$$

Equations (10), (11), (8), (9) hold for $k=1,\ldots,K$; $m=1,\ldots,M$; $j=1,\ldots,J$

Our task now is to solve these non-linear simultaneous equations for the equilibrium flow pattern $\{x_{jm}\}$, $\{\bar{x}_m^k\}$. We will accomplish this task by applying the Fixed Point algorithm of Scarf.

3. ON THE DETERMINATION OF AN APPROXIMATE FIXED POINT

3.1 Scarf Algorithm Summarized

In this section, we will present a restatement of Brouwer's fixed point theorem, and a constructive proof thereof developed by Scarf. The computation of the fixed point in the proof forms the basis of Scarf algorithm so that the reader should get a fairly good idea of the nature of a Scarf algorithm from this exposition.

The relevance of fixed points

Perhaps it would be illuminating to discuss the Walrasian model of pure-exchange economy in order to motivate the use of fixed points.[7] A fixed point of a mapping $y = f(x)$ is a point $\hat{x}$ such that $\hat{x} = f(\hat{x})$, i.e., a point that maps into itself.

Let $x = (x_1, \ldots, x_n)$ represent the (non-negative) prices of commodities $1, \ldots, n$, and let the excess demands at this vector of prices be represented by the continuous functions $g_1(x), \ldots, g_n(x)$ which are assumed to satisfy Walras law, a law derived from a "budget constraint".

$$\sum_{i=1}^{n} x_i g_i(x) = 0$$

A vector $\hat{x}$ is said to be an equilibrium price vector if all excess demands are less than or equal to zero at this price vector, i.e.,

$$g_i(\hat{x}) \leq 0, \quad i=1,\ldots,n$$

The computation of an equilibrium price vector is quite a difficult task to perform. One way of solving the problem is to transpose it into a problem of computing a fixed point, which can then be solved efficiently by the use of a Scarf algorithm. Let us postulate a mapping and show that its fixed point is the equilibrium price vector $\hat{x}$. Consider

$$y_i = \frac{x_i + \max[0, g_i(x)]}{1 + \sum_l \max[0, g_l(x)]}$$

We claim that the fixed point of this mapping is the vector $\hat{x}$. [At this point we will not toil over the proof of the existence of such a fixed point. We will simply assume that it exists.] A fixed point x^* of the above mapping satisfies

$$x_i^* = \frac{x_i^* + \max[0, g_i(x^*)]}{1 + \sum_l \max[0, g_l(x^*)]}$$

or

$$x_i^* \sum_l \max[0, g_l(x^*)] = \max[0, g_i(x^*)]$$

If $\sum_l \max[0, g_l(x^*)]$ is in fact greater than zero, the above equation implies that $g_i(x^*) > 0$ for every i with $x_i^* > 0$. Since all $x_i^* \geq 0$ and some are strictly positive, this violates Walras law. We conclude that $\sum_l \max[0, g_l(x^*)] = 0$ and therefore, $g_i(x^*) \leq 0$ $i = 1, \ldots, m$ hence x^* is an equilibrium price vector.

Brouwer's fixed point theorem

Let $y = f(x)$ be a continuous mapping of the simplex into itself. Then there exists a fixed point of the mapping, i.e., a vector $\hat{x}$ such that $\hat{x} = f(\hat{x})$.

Before we start the proof of this theorem, the concept of a primitive set needs to be introduced.

Definition

Given any list of vectors $x^{n+1}, \ldots, x^k$ in the simplex S the (n-m) vectors $x^{j_1}, \ldots x^{j_{n-m}}$, along with the m sides $s^{i_1}, \ldots, s^{i_m}$ form a primitive set if no vector $x^{n+1}, \ldots, x^k$ is interior to the simplex defined by $x_{i_1} > 0, \ldots, x_{i_m} \geq 0$ and

$$x_i \geq \min[x_i^{j_1}, \ldots, x_i^{j_{n-m}}] \quad \text{for} \quad i \neq i_1, \ldots, i_m$$

Note that the vectors in the list are indexed $(n+1), \ldots, k$ because the indices $1, \ldots, n$ are reserved for the sides of the simplex S; that is, $x^i, i = 1, \ldots, n$ refers to the $i^{\underline{th}}$ side of S. We now state an important lemma of Scarf's.

Scarf's lemma [7]

Let each vector in the list $x^1, \ldots, x^{n+1}, \ldots, x^k$ be labeled with one of the first n integers. Let x^j (for $j = 1, \ldots, n$) be given the label j. Then there exists a primitive set each of whose vectors has a different label.

Now recall that a vector x is in the simplex S if

$$x_i \geq 0 \qquad i=1,\ldots,n \tag{12}$$

and

$$\sum_{i=1}^{n} x_i = 1 \tag{13}$$

Thus, the requirement that $y = f(x)$ be a mapping from the simplex into itself implies that

$$\sum_{i=1}^{n} [f_i(x) - x_i] = 0 \tag{14}$$

It is clear that there is at least one i such that

$$f_i(x) \geq x_i \tag{15}$$

Label each vector x^j $(j = n+1, \ldots, k)$ in the following manner:

$$\begin{aligned} &\text{label} \quad (x^j) = i_j \\ &\text{where} \quad i_j = \min\{l \mid f_l(x^j) \geq x_l^j\} \end{aligned} \tag{16}$$

The vectors $x^j (j=1,\ldots,n)$ are labeled j. Now, the conditions of Scarf's lemma are satisfied, and hence there exists a primitive set whose labels are all different. That is, there exists a primitive set $(x^{j_1},\ldots,x^{j_n})$ such that

$$f_{i_j}(x^j) \geq x_{i_j}, \quad j=j_1,\ldots,j_n \tag{17}$$

where $i_{j_1},\ldots,i_{j_n}$ are all distinct from each other.

Let us now demonstrate Brouwer's theorem by taking a finer and finer collection of vectors which, in the limit, become everywhere dense on the simplex. Each such collection will determine a geometric subsimplex with the above property. As the vectors are increasingly refined, a convergent subsequence of subsimplices may be found, which tend in the limit to a single vector x*. From the continuity of the mapping the vector x* must have the property that

$$f_i(x^*) \geq x_i^* \qquad i=1,\ldots,n \tag{18}$$

But (14) holds for any x, and in particular for x*.

$$\sum_{i=1}^{n} [f_i(x^*) - x_i^*] = 0 \tag{19}$$

(18) and (19) imply that

$$f_i(x^*) = x_i^* \qquad \text{for all } i \tag{20}$$

demonstrating Brouwer's theorem.

It is a fact that we cannot really go to the limit in an actual application on a computer. But the final primitive set with distinct labels could be averaged out and the resultant vector becomes an approximation of the true fixed point. Furthermore, we can make the approximation as good as we desire by simply taking a fine enough collection of vectors.

This development is the spirit of Scarf's algorithm for computing approximate fixed points. More specifically, to use Scarf algorithm to compute the fixed point of any continuous mapping from the simplex into itself, we must specify the following:

— A finite list of vectors in the simplex
— A Labeling Procedure
— Replacement Operation
— Final Termination Routine

The algorithm then procedes as follows. Each of the vectors in the list is labeled according to the specified labeling procedure. An initial primitive set is created and a check is made to see if each of the members has a distinct label. If such is not the case, the algorithm constructs a new primitive set in a manner specified in the replacement operation and repeats the check to find out if the new primitive set is "completely-labeled". The process is continued until a completely-labeled primitive set, whose existence is guaranteed by Scarf's lemma, is obtained. The final termination routine then averages out the vectors in the final primitive set to give a good approximation of the fixed point.

Note that the labeling procedure is determined by what mapping is being considered whereas the creation of the list of vectors, the specification of the replacement operation and the final termination routine rely only peripherally on the specific mapping under investigation.

3.2 Computation of the Equilibrium Flow Pattern - An Example

A seemingly restrictive assumption that needs to be satisfied if we were to apply Scarf algorithm is the condition that $y=f(x)$ be a mapping from a simplex into itself. However, we can define a suitable artificial mapping from the simplex into itself the property that its fixed point corresponds to the desired quantity which, in this case, is the equilibrium flow pattern of a transportation system.

For the sake of exposition, let us, at this point, formulate a simple model. Consider the case where no equilibration needs to be done on the access components. Hence $\bar{x}_m^k$ are fixed constants.

Define

$$\tau_{jm} = \frac{x_{jm}}{MJ\bar{t}_j} \qquad j=1,\ldots,J;\ m=1,\ldots,M \tag{21}$$

and

$$\tau_{om} = \frac{1}{M} - \sum_{j=1}^{J} \tau_{jm} \qquad m=1,\ldots,M \tag{22}$$

where $\bar{t}_j$ is the upper limit of all x_{jm}:

$$x_{jm} \in [0,\bar{t}_j] \qquad j=1,\ldots,J \tag{23}$$

Clearly the "vectors" τ_{jm} are in the simplex:

$$\tau_{jm} \geq 0 \qquad j=0,1,\ldots,J\ ;\ m=1,\ldots,M$$

and

$$\sum_{j=1}^{J} \sum_{m=0}^{M} \tau_{jm} = 1$$

The assumption of no equilibration on access implies that we have determined all variables in the system defined by (8), (9), (10), (11) except for $[x_{jm}]$.

It is easy to verify that the following transformation satisfies the condition of Brouwer's theorem and has a fixed point that corresponds to an equilibrium vector $[x_{jm}]$ for our above example.

$$G_{jm}(\tau) = \frac{1}{MJ\bar{t}_j} S_{jm}(D_{j1}(\tau),\ldots,D_{jM}(\tau))$$

$$G_{om}(\tau) = \frac{1}{M} - \sum_{j=1}^{J} G_{jm}(\tau) \tag{24}$$

where

$$D_{jm}(\tau) = \frac{1}{\varepsilon} \sum_{k=1}^{K} \delta_{kj} \frac{\exp[b(\sum_{j=1}^{J} \delta_{kj} MJ\bar{t}_j \tau_{jm}) + b\bar{x}_m^k + ay_m^k]}{\sum_{l=1}^{M} \exp[b(\sum_{j=1}^{J} \delta_{kj} MJ\bar{t}_j \tau_{jl}) + b\bar{x}_l^k + ay_l^k]}$$

$$j=1,\ldots,J \qquad m=1,\ldots,M \tag{25}$$

The fixed point τ^* of the transformation defined by Equations (24) and (25) have the property that

$$G_{jm}(\tau^*) = \tau^*_{jm} \qquad j=0,\dots,J\ ;\ m=1,\dots,M \tag{26}$$

Associated with each τ^*_{jm} is a unique x^*_{jm}

$$x^*_{jm} = \tau^*_{jm} MJ\bar{t}_j \qquad j=1,\dots,J;\ m=1,\dots,M \tag{27}$$

It should be observed that at segment travel times x^*_{jm}, $j=1,\dots,J$; $m=1,\dots,M$, each and every worker in our sample plans his travel in such a way that the segment demands are

$$D_{jm}(x^*) = \frac{1}{\theta}\sum_{k=1}^{K}\delta_{kj}\,\frac{\exp[b(\sum_{j=1}^{J}\delta_{kj}x^*_{jm} + b\bar{x}^k_m + ay^k_m]}{\sum_{l=1}^{M}\exp[b(\sum_{j=1}^{J}\delta_{kj}x^*_{jl} + b\bar{x}^k_l + ay^k_l]} \qquad j=1,\dots,J;\ m=1,\dots,M$$

The travel times "supplied" by the transportation system in response to these demands, are, in turn, given by:

$$S_{jm}(D_{j1}(x^*),\dots,D_{jM}(x^*)) \qquad j=1,\dots,J;\ m-1,\dots,M$$

which, in view of Equations (24), (26) and (27), turn out to be exactly

$$x^*_{jm} \qquad j=1,\dots,J;\ m=1,\dots,M$$

Hence, as soon as we have $[\tau^*_{jm}]$, the equilibrium flow pattern $[x^*_{jm}]$ obtains immediately.

Now we apply Scarf's fixed point algorithm to compute $[\tau^*_{jm}]$. By specifying a grid of vectors and utilizing a labeling procedure similar to (16), we get out of the algorithm a final primitive set, each of whose members is "close" to $[\tau^*_{jm}]$. To get a good approximation of $[\tau^*_{jm}]$, we average out the members of the final primitive set in the manner outlined by Shoven in Appendix A of [9]. From the discussion in the preceding section, it is clear that we can get as good an approximation of $[\tau^*_{jm}]$ as we desire by simply making the grid of vectors fine enough. Thus, Scarf algorithm can give us an approximation of the equilibrium flow pattern $[x^*_{jm}]$ which can be made as good as desired.

4. EMPIRICAL APPLICATION

We are currently in tre process of applying our equilibrium model to the I-580 corridor in the San Francisco Bay Area. The I-580 model has the following characteristics:

1. A maximum of 8 modes are available to each worker in the sample.
2. The single-bottleneck formulation is used to characterize the supply equations.
3. There are 9 segments in the corridor.

Although we do not have at hand a final result of this study, initial runs of the model seem very encouraging. Without the Final Termination Routine, it costs less than \$ 70 to run the model. Since the Final Termination Routine is basically a Simplex algorithm, we wouldn't expect the cost to go up too much by incorporating it into the model. Far from being conclusive, our experience should nonetheless indicate the magnitude of the cost of implementing our model. We should add that our current programs are by far *not* the most efficient possible, and there is room for further improvements. We will report our final results in a forthcoming paper.

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Dan McFadden originally suggested that we should look into formulating the equilibration of mode choices as an equilibrium rather than a maximization problem and indicated that the economist's experience with Scarf algorithm has been good; he also suggested that in order to reduce the number of "markets" to be equilibrated in a corridor study, the "markets" could be defined as the segments of this corridor rather than the customary zonal interchanges. The present working paper is the result of following up these suggestions.

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REFERENCES

[1] Dafermos, S. D., and F. T. Sparrow, "The Traffic Assignment Problem for a General Network", **Journal of Research,** National Bureau of Standards, Vol 37-B, No. 2, 1969.

[2] May, A. and H. Keller, "A Deterministic Queueing Model", Transportation Research, p. 117-127, 1967.

[3] McFadden, D., "Conditional Logit Analysis of Qualitative Choice Behavior", in P. Zarembka, editor, **Frontiers in Econometrics,** Academic Press, New York, 1973.

[4] McFadden, D., "The Measurement of Urban Travel Demand", **Journal of Public Economics,** 1974.

[5] Nguyen, S., "A Unified Approach to Equilibrium Methods for Traffic Assignment", paper presented at the International Symposium on Traffic Equilibrium Methods, Montreal, 1974.

[6] Scarf, H., "The Computation of Economic Equilibria", Cowles Foundation Monograph no. 24, 1973.

[7] Scarf, H., "The Approximation of Fixed-Points of a Continuous Mapping", **SIAM Journal of Applied Mathematics,** 15:1328-43, 1967.

[8] Shoven, J. and J. Whalley, "On the Computation of Competitive Equilibrium in International Markets with Tariffs", **Journal of International Economics,** November 1974.

[9] ......, "A General Equilibrium Calculation of the Effects of Differential Taxation of Income From Capital in the US", **Journal of Public Economics,** p. 281-321; 1972.

[10] Small, K., "Bus Priority, Differential Pricing and Investment in Urban Highways", **Working Paper 7613,** Urban Travel Demand Forecasting Project, Institute of Transportation Studies, University of California, Berkeley, 1976.

[11] Talvitie, A. and Y. Dehghani, "Supply Model for Transit Access and Linehaul", **Working Paper 7614,** Urban Travel Demand Forecasting Project, Institute of Transportation Studies, University of California, Berkeley, 1976.

[12] Wardrop, J. G., "Some Theoretical Aspects of Road Traffic Research", **Proceedings of the Institute of Civil Engineers, Part III,** Vol. 1, 1952.

FOOTNOTES

1) By equilibrium flow pattern we mean a complete list of *mode splits* on *all* levels of aggregation. From individual paths up to zonal or segment (a conglomeration of zones) volumes.

2) Actually, since equilibration is done on persons rather than vehicles, in this paper, to be consistent, γ^1_2 should really be "the number of car-equivalents of an Express Bus divided by the number of passengers".

3) The express bus travel time, to be more realistic, must be increased by the time needed for stops. This time is a function of stop spacing and volume entering and exiting the bus. The latter board/alight time is represented as a constant rather than func-

tion, in a present application of the model. Also note that part of the auto volume can be diverted to arterials (assuming Wardrop's principle) without making the model more complex. For simplicity of presentation these details are omitted here.

4) Some of the readers may feel uneasy about such a simple model of highway network performance. They are reminded that the current network algorithms also consider every link independently, "load" the zones in one spot into the network and, thus, differ only marginally from the multiple bottleneck version here.

5) "Access" will, from now on, stand for both access and egress.

6) We should really write "vehicle or person", since in guideway modes persons are the basic units. However, we feel that equilibration will be confined to the highway and not guideway modes. Hence, we describe our model in a way that ignores equilibration on the latter modes. It should be clear, though, that our presumption imposes no restriction on the model's generality.

7) The following example is borrowed from Scarf [6] .