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MODELLING

A RAMSEY PRICE EQUILIBRIUM MODEL FOR URBAN TRANSIT SYSTEMS: A BILEVEL PROGRAMMING APPROACH WITH TRANSPORTATION NETWORK EQUILIBRIUM CONSTRAINTS

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Abstract

In this paper we present the variational inequality formulation of the Ramsey price equilibrium, being a natural extension for nonlinear programming formulation to that problem, and a computation algorithm based on the sensitivity analysis for the restricted variational inequality.

INTRODUCTION

Many models have been proposed so far to determine optimal transportation systems operations or strategic plans which take into account both supplier and user behavior. These include the network design problem [Dafermos (1968), Abduaal and LeBlanc (1979), Marcotte (1983), Sasaki and Asakura (1987), LeBlanc and Boyce (1986), Kim (1990), Friesz et al. (1990)], the signal optimization problem [Tan et al. (1979), Marcotte (1983), Fisk (1986)], the traffic control problem on general traffic corridor systems (Yang and Yagar 1994) and the optimal pricing in transit systems (Kawakami and Mizokami 1985; Fisk 1986; Miyagi et al. 1992). These problems have the common structure of a hierarchical decision-making process: a single decision-making unit which may represent a supplier of transportation services or controlling agency tries to make optimal decisions with respect to the transportation system, which may restrict the feasible constraints set for user of that transportation system and influence user behavior or demands. Algebraically, the decision making problem mentioned above is typically described as the following *bilevel programming problem* or *Stackelberg problem* (eg Bard 1983; Shimizu 1982):

[P]

U) $\min.\ f_1(x,y(x)),\ \text{s.t. } g_1(x,y(x)) \leq 0$

where y(x) is defined as the solution for

L) $\min.\ f_2(x,y), \text{s.t. } g_2(x,y) \leq 0$

where f_1, f_2 are objective functions and g_1, g_2 constraint functions, respectively, in which decision variable vectors are x in U and y in L. U) is defined as a upper problem (or a leader, policy problem) and L) a lower problem (or a follower, behavioral problem) (see Candler and Townsley 1983). In this paper we call [P] an *applied network equilibrium analysis* (or an ANE model) when the lower problem consists of user equilibrium models in transportation networks.

Sensitivity analysis approaches have received significant attention in recent years as the most important method for developing solution algorithms for the bilevel programming problem with user-equilibrium constraints. The ANE model requires methods to approximate numerically a new equilibrium solution resulting from a change of decision variable vector in the upper problem. For this purpose, the sensitivity analysis for the restricted variational inequality problem developed by Tobin and Friesz (1988) is of special importance because it can provide the calculation of derivatives of decision variables and constraints multipliers with respect to perturbation parameters in the lower network equilibrium problem, thus, it enables us to provide the information associated with the derivatives of the objective function in the upper problem. Kim (1990) proposed a new approach to the optimal network design problem based on the nonlinear sensitivity analysis by Fiacco (1983). Friesz et al. (1990) applied the sensitivity analysis by Tobin and Friesz (1988) to the development of heuristic algorithm for the network design problem. Yang and Yagar (1994) also applied the sensitivity analysis for nonlinear programming to the integration problem of traffic assignment and traffic control on general traffic corridor systems. Kim in association with Suh (1990) provides almost a complete list for applications of bilevel programming approach and solution algorithms proposed so far.

The main purpose of this paper is to provide a bilevel programming model for determining an optimal pricing in a transit system so as to maximize the social welfare compatible to user equilibrium on a multimodal network and to give a computational procedure for the bilevel programming model based on a perturbed variational inequality problem for the restricted problem.

In our ANE model, the upper problem is defined as the maximization problem of the sum of consumer and producer surpluses where while the consumer surplus is defined by the expected maximum utility derived from the random utility theory, the producer surplus is given by the usual profit-maximization behavior subject to a break-even constraint. The provider for transit service is assumed to give different transportation services using common costs and to be able to give a different price in each service. In this situation the Ramsey rule is directly applicable for determining the optimal price for each service in a sense that total surplus consisting of consumer surplus plus firm profit results in the smallest loss in surplus, compared with the "first best" pricing. An application of the Ramsey rule to transit pricing has been conducted by Train (1977) for AC transit and BART, however, a competition through pricing mechanism between transit and auto is not taken into account in this application. Ramsey pricing should be examined within the framework of the multimodal network equilibrium because there exists mutual interaction between each transportation service network with changing prices of each service and as well as travel time arising from the usage of road network. Thus, the lower problem in our ANE model must consist of the multimodal equilibrium problem. This type of ANE model was first proposed by Miyagi et al. (1992), in which a nonlinear optimization formulation for multimodal equilibrium is adopted, and is called a *Ramsey price equilibrium problem.*

In this paper we present the variational inequality (VI) formulation of the Ramsey price equilibrium, being a natural extension of nonlinear programming formulation to that problem, and a computation algorithm based on the sensitivity analysis for the restricted variational inequality by Tobin and Friesz (1988). This paper considers a feasible direction method to solve the Ramsey price equilibrium formulated as a bilevel programming problem. The similar problem as the ANE model considered here is also studied by Fisk (1986). In Fisk's approach, however, the upper problem is formulated as a maximizing a profit of transit firm and a penalty function approach like Simizu and Aiyoshi (1981) is adopted for solving for bilevel programming problem.

THE RAMSEY PRICING RULE FOR TRANSIT SYSTEMS

If a multiproduct firm is a natural monopoly, then pricing goods at their marginal cost can result in the firm losing money. If the firm cannot be subsidized, for the firm to be sustainable, the firm must set prices sufficiently above marginal cost to break-even, that is, earn zero profit. In a one-good situation, the requirement of zero profit is sufficient to set prices equal to average cost. However, with more than one good, many different price combinations result in zero profit. Ramsey (1927) first addressed this kind of problem in the context of optimal taxation and developed a method for determining the tax rates for various goods that would provide the government with sufficient revenue while reducing consumer surplus as little as possible. Because the break-even constraint prevents the imposition of a fully optimal, the so-called "first-best", marginal cost prices, we refer to prices which maximize total surplus subject to breaking even as "second-best" prices. Baumol and Bradford (1970) have pointed out that optimal taxation rules proposed by Ramsey are directly applicable for determining second-best prices for multiproduct natural monopolies.

Possible situations where the Ramsey rule is applicable to the pricing of transit service are discussed here.

(i) Most public transit providers are natural monopolies in that their marginal cost is below their average cost over the relevant range of output.

(ii) A transit system operator usually provides more than two different services: two modes like bus and street car, different routes with various service frequencies.

(iii) Since providers of transit services cannot be fully subsidized, they may have to bear the burden of covering the shortage of fixed costs. This implies that the first-best pricing would result in the providers facing negative profit and not able to continue to operate in the long run.

(iv) A regional transportation agency must coordinate service among the various transit agencies and exercise considerable oversight of each agency's fare. From the social-welfare point of view, it may be justifiable for the regional transportation agency to adopt the total surplus maximization approach.

Let us first confirm our attention to a simple pricing problem in which social welfare measured by total profits plus consumer surplus is maximized subject to a break even constraint. That is, as usual, it is assumed that profits are distributed among consumers. If the individual indirect utility functions are linear in income with the same marginal utility of income, then the way profits are shared dose not affect the level of social welfare nor demands. Symbolically, the most efficient uniform second-best prices are given by

$$[U0] \qquad \max_{P} \Pi_1 (p; q) = CS(p) + PS(p), \quad \text{s.t. } PS(p) \le K \qquad (1)$$

where CS and PS represent consumer surplus and producer surplus, respectively, and K the money transfer from the government. Given a set of prices $p \in R^m = [p_1, p_1, \cdots, p_m]$, where p_i denotes a price for the service i and m represents the number of service, the consumer surplus can be written as the line integral

$$CS(p) = \int_p^{\infty} \sum_{i=1}^{m} q_i(p) \ dp_i \qquad (2)$$

and the producer's surplus is given by

$$PS(p) = \sum_{i=1}^{m} p_i q_i(p) - T(q_1(p), q_2(p), \cdots, q_m(p)) \qquad (3)$$

where $q_i(p)$ is demand function for service i and T(q) is the joint cost function of the transit firm. The corresponding conditions that the optimal solution should satisfy are:

$$\sum_i (p_i - MC_i) \frac{\partial q_i}{\partial p_j} = - \frac{\lambda}{\lambda+1} q_j \qquad , j=1,2,\cdots,m \qquad (4)$$

where $MC_i = \frac{\partial T}{\partial q_i}$ is marginal cost of service i. From the integrability conditions for (2), $\left\{ \frac{\partial q_i}{\partial p_j} = \frac{\partial q_j}{\partial p_i} \right\}$, (4) can be transformed as :

$$\sum_i \frac{p_i - MC_i}{p_i} \varepsilon_{ji} = - \frac{\lambda}{\lambda+1} = -\beta \qquad (5)$$

where ε_{ji} is price elasticity of demand for service j with respect to the price of service i. To give some idea of how the Ramsey rule looks, let us consider the two service case. In that case, it follows from (5) that

$$\frac{p_1 - MC_1}{p_1} (\varepsilon_{11} - \varepsilon_{21}) + \frac{p_2 - MC_2}{p_2} (\varepsilon_{12} - \varepsilon_{22}) = 0 \qquad (6)$$

If cross-elasticity are zero, (6) is further simplified and results in the well-known Inverse Elasticity Rule, or IER, viz.,

$$\frac{p_1 - MC_1}{p_1} \varepsilon_{11} + \frac{p_2 - MC_2}{p_2} \varepsilon_{22} \qquad (7)$$

BINARY MODE CHOICE / ASSIGNMENT MODEL

To construct a model of hierarchical systems in general, behavior at the lowest level (transportation system users) is modeled first and higher levels are added in steps. Now we will show the behavioral model at the lower level in a bilevel system.

The expanded network is represented by a directed graph G(N, A) where N is the set of nodes and A is the set of directed links. A subset of nodes serve as origins and/or destination for trips. The set of all origin / destinations (O/D) pair is designated by I. The network permits the flow of vehicles and transit passengers on links. The transit vehicles follow fixed itineraries. The nodes n, $n \in N$, represent origins, destinations and intersections of links; the links a, $a \in A$, represent the road and transit infrastructure of the urban area. Each link may be accessible to private and transit vehicles. The modes are designated by index m which is 1 for the automobile mode and 2 for the transit mode. The individual user cost c_a^m for travel by mode m on link a are given by well-defined functions of the link flow vector of both modes, v_a^1 and v_a^2, but are separable by link as follows:

$$c_a^m = c_a^m(v_a),\ m=1,2 \text{ where } v_a = (v_a^1, v_a^2),\ a \in A \tag{8}$$

The user cost functions, $c_a(v_a)=[c_a^1(v_a), c_a^2(v_a)]$, are assumed to be monotone, continuous and differentiable:

$$\left(c_a(v'_a)-c_a(v''_a)\right)^T(v'_a-v''_a) \geq 0,\ a \in A \tag{9}$$

The origin to destination demands, q_i^m, $i \in I$, for each mode m may use directed paths k, $k \in \Lambda_i^m$, where Λ_i^m is the set of paths, ($\Lambda_i^m \neq \phi$), available for mode m and O/D pair i. The total origin to destination demands by both modes, $q^1 \in R^{IxI}$ and $q^2 \in R^{IxI}$, are given by a constant matrix $\overline{q} \in R^{IxI}$, where for OD conservation equation is given as :

$$q_i^1 + q_i^2 = \overline{q_i},\quad i \in I \tag{10}$$

The flows on paths k, h_k, satisfy conservation of flow and nonnegativity.

$$\sum_{k \in \Lambda_i^m} h_k = q_i^m,\ i \in I,\ m=1,2 \quad \text{and } h_k \geq 0 \tag{11}$$

The conservation equations (10) and (11) can be described using OD pair-path incident matrix Λ as follows:

$$\Lambda h = q,\ h \geq 0 \tag{12}$$

where $\Lambda \in R^{2IxK}$ $(K = k_1+k_2)$ is defined by the OD pair-path incidence matrices $\Lambda^1 \in R^{k_1}$, $\Lambda^2 \in R^{k_2}$ by each mode

$$\Lambda = \begin{bmatrix} \Lambda^1 & 0 \\ 0 & \Lambda^2 \end{bmatrix}$$

and $q \in R^{2I}$ is the OD trip vector represented solely by the auto trip demand:

$$q = [q_1, q_2, \cdots, q_i, \cdots, q_I, \overline{q_1}-q_1, \cdots, \overline{q_i}-q_i, \cdots, \overline{q_I}-q_I\ ]^T$$

The link flows v_a^m are given by

$$v_a^m = \sum_{i \in I} \sum_{k \in \Lambda_i^m} \delta_{ak}\, h_k,\ m = 1, 2,\ a \in A$$

where

$$\delta_{ak} = \begin{Bmatrix} 1 \text{ if link a belongs to path k} \\ 0 \text{ otherwise} \end{Bmatrix}$$

Correspondence between link and path variables can be made using the link-path incidence matrix Λ. We have

$$v = \Delta h \quad (13)$$

The sets of feasible flow corresponding to link flow formulation and path flow formulation respectively are then defined as:

$$\Omega' = \left[v\text{: } v = \Delta h,\ q = \Delta h,\ h \geq 0 \right] \quad (14a)$$

$$\Omega = \left[h\text{: } q = \Delta h,\ h \geq 0 \right] \quad (14b)$$

The cost of each path $C_k(v)$ is the sum of the user costs of the links in the path

$$C_k(v) = \sum_{a \in A} \delta_{ak} c_a^m(v_a),\ k \in \Lambda_i^m,\quad i \in I,\ m=1,2 \quad (15)$$

or

$$C(v) = \Lambda^T c(v)$$

Let $u_i^m(v)$ be the cost of the least cost path for all O/D pairs i and modes m

$$u_i^m(v) = \min.\ k \in \Lambda_i^m\ C_k(v),\quad i \in I,\ m=1,2 \quad (16)$$

We now consider the case where a mode choice function $G_i(w_i)$ is a probabilistic choice function such as the logit model, that depends on the travel costs by the two modes via their difference w_i, $w_i = u_i^1(v) - u_i^2(v)$. For each centroid pair, one can eliminate the transit demand using (10) and it is possible to derive a function G so that the auto demand can be obtained from $q_i^1 = \overline{q_i} G_i(w_i)$. It is assumed that $G_i(w_i)$ is a strictly decreasing function with inverse $W_i\left(q_i^1/q_i\right)$. Since $\overline{q_i}$ is constant for a given i we refer to the inverse function as $W_i(q_i^1)$.

The binary mode choice/assignment model is formulated by supposing that no traveler has the incentive to change mode

$$u_i^{1*} - u_i^{2*} = W_i(q_i^{1*}),\quad i \in I \quad (17a)$$

and that for each mode the path choice satisfies Wardrop's user optimized behavioral principle

$$C_k^* - u_i^{m*} \begin{cases} =0 \text{ if } h_k^* > 0, \\ \geq 0 \text{ if } h_k^* = 0, \end{cases} k \in \Lambda_i^m,\quad i \in I,\quad m=1,2 \quad (17b)$$

subject to the feasibility constraints (14a). The notation (*) indicates equilibrium values of the flows, demands and O/D costs. The equilibrium conditions for the combined mode choice and assignment problem, (17) can be expressed as the VI:

[L0] For path flow formulation,

$$c\left(h^*\right)^T \left(h - h^*\right) - W\left(q^{1*}\right)^T \left(q^1 - q^{1*}\right) \geq 0,\quad \forall (v, q) \in \Omega \quad (18a)$$

and for link flow formulation,

$$c\left(v^*\right)^T \left(v - v^*\right) - W\left(q^{1*}\right)^T \left(q^1 - q^{1*}\right) \geq 0,\quad \forall (v, q) \in \Omega' \quad (18b)$$

A mode choice function $G_i(w_i)$,which satisfies rational assumptions about the behavior of travelers in choosing their mode, is a strictly decreasing function of w_i. We assumed that this property holds, that is,

$$\text{for } w' \neq w'', \ \left(G(w') - G(w'')\right)^T \left(w' - w''\right) < 0 \tag{19}$$

This implies that -G(w) is a strictly monotone mapping. We recall that the user cost functions were assumed to be monotone, as stated in (9). When the link user cost functions are strictly monotone and the mode choice functions are strictly decreasing functions of the difference in travel costs, the demands, link travel costs and O/D travel costs are unique. The verification of monotonicity conditions (19) is trivial for logit mode choice functions. Florian and Spiess (1983) shows that if $c_a^1(v_a^1) = c_a^1(v_a^1)$ and $c_a^2(v_a) = c_a^2(v_a^2)$, then the solution of (18b) is equivalent to

[LO′]

$$\text{Min.}Z = \sum_{a\in\Lambda} \int_0^{v_a^1} c_a^1(x)dx + \sum_{a\in\Lambda} \int_0^{v_a^2} c_a^2(x)dx - \sum_{i\in I} \int_0^{q_i^1} W_i(y)dy \tag{20}$$

subject to (14a).

RAMSEY PRICE EQUILIBRIUM MODEL

Transportation supplier at the upper level controls level of service characteristics and fares or rates. Userresponses to supplier decisions are included in the model as constraints to ensure that the solution is optimal after users have reached an equilibrium compatible with these decisions. In this paper only fare of each route of public transportation system is taken into account as a control parameter. Ramsey price equilibrium with logit demand function now can be described by the leader-follower problem considered by Stackelberg (1934), where while a leader determines prices of each route of transit mode by the Ramsey pricing rule given the share of each mode and network flows, a follower determines equilibrium flows given a set of prices. We first show the consumer surplus function corresponding the logit modal share function, then the formal mathematical structure for Ramsey price equilibrium problem for transit systems.

Indirect utility function for logit model

Suppose that there are m+1 commodities and q statistically identical and independent consumers. Commodity 0 is perfectly divisible and is taken as the numeraire. Commodities i = 1...m are the transportation services with prices $p_1 \ldots p_m$ and with quality indices $a_1 \ldots a_m$. Each consumer has the same (real) income y and travels with any one of transportation services. Assume that $0 < p_i < y$, i =1...m, to ensure that each service can be afforded by all consumers. Furthermore assume that a consumer's conditional indirect utility from using service i is given by the additive form

$$\tilde{V}_i = y - p_i + a_i + \varepsilon_i, \quad i=1\cdots m, \tag{21}$$

where the e_i are independent and identical double exponential variables. Providing the fixed total demand $\bar{q}$ is given, the expected demands are then given by

$$q_i = \bar{q}\, \frac{\exp\left[(a_i - p_i)/\theta\right]}{\sum_{j=1}^{m} \exp\left[(a_j - p_j)/\theta\right]}, \quad i=1\cdots m \tag{22}$$

Then, we obtain an explicit form for the indirect utility (consumer surplus) function of the representative consumer;

$$V = Y + \bar{q}\theta \ln\left[\sum_{j=1}^{m} \exp\left(\frac{a_j - p_j}{\theta}\right)\right] \tag{23}$$

where Y is the aggregate income of each consumer (Anderson et al. 1993).

Formulation

We formulate Ramsey price equilibrium problem (RPEP) with the application of the binary choice/assignment model described in the previous section, but, with a slightly different way. Figure 1 shows the conceptual network configuration taken into consideration here.

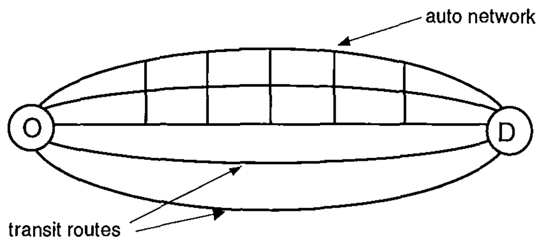

Figure 1 Multimode network representation for a single OD pair

Mutual exclusive transit networks are assumed to handle a multimodal network equilibrium problem within the context of the binary modal choice formulation. While demand shares between private automobile and public transit systems are determined by logit modal split functions, demand for each public transit available is assumed to be determined as the path flow on each public all transit route with user equilibrium mechanism. The break-even constraints are imposed on transit routes.

Algebraically, RPEP can be written as follows:

U1)

$$\text{Max.}\Pi(h, p) = \theta \sum_{i \in I} \overline{q_i} \ln \sum_{m} \exp\left[\frac{(-u_i^m)}{\theta}\right] + \sum_{i \in I} \sum_{k \in \Lambda_i^2} [p_k h_k - T_k(h(p))]$$

$$\text{s.t.} \sum_{i \in I} \sum_{k \in \Lambda_i^2} (p_k h_k - T_k(h(p))) \le K$$

L1) (q,h) is the solution for the VI :

$$c(h^*)^T (h-h^*) - W(q^{1*})^T (q^1 - q^{1*}) \ge 0, \quad \forall (v, q) \in \Omega$$

where u_i^m represent the generalized cost for traveling between OD pair $i \in I$ using private auto or public transit, being defined as $u_i^m = p_i^m + b^m t_i^m$ in which p_i^m, t_i^m, are prices and travel times of mode m (m = 1,2). Since p_i^m for auto mode are assumed fired and constant, we treat only prices for transit modes and denote these p_k (because transit routes k correspond to transit modes in our formulation).

Let (q1*, h*) be a solution vector to the VI[L1]. Then it follows by Theorem 1 in Tobin and Friesz (1988) that the necessary conditions for solving [L1] are:

$$-w(q^{1*}) - \varphi + E\mu = 0 \tag{24a}$$

$$C(h^*) - \lambda - \Lambda^T \mu = 0 \tag{24b}$$

$$\varphi^T q^* = 0 \tag{24c}$$

$$\lambda^T h^* = 0 \tag{24d}$$

$$\Lambda h^* - q^* = 0 \tag{24e}$$

$$\varphi \geq 0, \lambda \geq 0 \tag{24f}$$

where $y \in R^I$, $l \in R^{k1+k2}$, $m \in R^{2I}$ are Lagrange multipliers and E is defined by unit matrix, e, with dimension I as E=[e -e]. The system of equation, however, does not meet the sufficient condition, as stated as Theorem 3 in Tobin and Friesz, for a local isolated minimizing point because of the non-uniqueness of the solution.

SENSITIVITY ANALYSIS FOR THE RESTRICTED VARIATIONAL INEQUALITY

If it is already known that a solution of nonlinear equation systems (24) exists and is uniquely determined, then a parametric optimal solution q* and Lagrange multiplier Ψ = [y, m, l] may be represented by implicit functions h(p), g(p) as price vector being independent variable (Simizu 1982). Then, RPEP can be furthermore simplified as:

$$\max_{P}.\ \Pi(p, \eta(p)), \qquad \text{s.t. } PS\ (p, \eta(p)) \leq K \tag{25}$$

Thus, once the derivatives of the lower level is obtained with respect to the decision variables of the upper level, many algorithms for the standard nonlinear programming problem can be utilized to solve the ANE model in the previous section. For this purpose, the nonlinear sensitivity analysis is useful because that any parameter perturbation will generally results in the network equilibrium solution and that this type of sensitivity analysis requires the calculation of decision variables and constraint multipliers with respect to perturbation parameters (Fiacco 1983; Tobin and Friesz 1988). The perturbed equilibrium network flow problem can be written as the following perturbed variational inequality (Tobin and Friesz 1988).

L2) Find $h^* \in W(\varepsilon)$ such that

$$C(h^*, \varepsilon)^T (h-h^*) - W(q^{1*}, \varepsilon)^T (q^1-q^{1*}) \geq 0 \tag{26}$$

for all h, q, where

$$\Omega(\varepsilon) = \left[h \mid \Lambda h = q(\varepsilon),\ h \geq 0\right] \tag{27}$$

and ε is a vector of perturbation parameters.

Since the path flows are not unique, this formulation will not satisfy Theorem 3 in Tobin and Friesz (1988), thus, derivatives of a solution h* with respect to the perturbation parameter do not exist. In order to resolve this problem, the approach taken in Tobin and Friesz is to select one particular path flow solution, in particular nondegenerate extreme point of the polytope defined as:

$$\Gamma^*(\varepsilon) = \left[h \mid \nabla h = v^*,\ \Lambda h = q(\varepsilon),\ h \geq 0\right]$$

Given the solution v*(0), the first step is to choose a unique path flow vector h* to associate with v*(0). The existence of unique path flow can be ensured by the flow decomposition principle (Ahuja, Magnanti and Orlin 1993) which may be described as that every nonnegative link flow can be represented as a path flow if every directed path with positive flow connects an origin node to a destination node. The only requirement in this choice is that h* be a nondegenerate extreme point of Γ*(0), that is, an extreme point solution h* included in the polyhedron Ω.

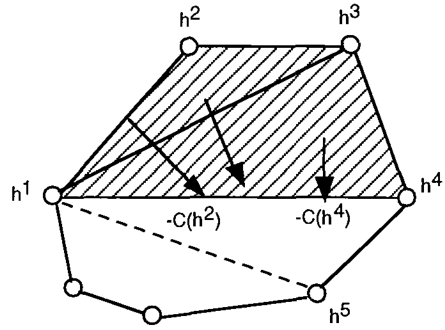

Figure 2 Generation of extreme point solutions in the polyhedron Ω

A number of path generating methods can be used to generate h*. One way to find h* is to solve the linear programming problem in which v* serves as the vector of capacities (Tobin and Friesz 1986; Ahuja et al. 1993).

Because only positive path flows are considered, nonnegativity constraints will be nonbinding at the optimal solution and remain so for the perturbation parameter in a neighborhood of 0. Thus, the equilibrium conditions (24) reduces to

$$-w^0\left(q^{1*},0\right)+E\mu=0 \tag{28a}$$

$$C^0\left(h^*,0\right)-\Lambda^{0T}\mu=0 \tag{28b}$$

$$\Lambda^0 h^*-q^*(0)=0 \tag{28c}$$

It can easily be seen that the columns of Λ^T are linearly independent and so μ is unique. Denote this unique vector as μ^*. Therefore, the conditions for Theorem 4 are satisfied by the system (28), and so the derivatives of h^{o*} with respect to ε may be calculated. The Jacobian matrix of the system (28) with respect to (h^o,μ) and evaluated at ε= 0 is

$$J_{q^o},h^0,\mu=\begin{bmatrix} -\nabla_g w^0(q^*,0) & 0 & E \\ 0 & \nabla_h C^0(h^*,0) & -\Lambda^{0T} \\ E^T & \Lambda^0 & 0 \end{bmatrix} \tag{29}$$

The Jacobian matrix of the system (29) with respect to ε and evaluated at zero is

$$J_\varepsilon=\begin{bmatrix} -\nabla_\varepsilon w^0(q^*,0) \\ \nabla_\varepsilon C^0(h^*,0) \\ -\nabla_\varepsilon q(0) \end{bmatrix} \tag{30}$$

Therefore the gradient vector of solution at the lower level with respect to the perturbation is obtained as:

$$\begin{bmatrix} \nabla_\varepsilon q^0 \\ \nabla_\varepsilon h^0 \\ \nabla_\varepsilon \mu \end{bmatrix}=J^{-1}_{q^0},h^0,\mu\,.\,J_\varepsilon \tag{31}$$

Suppose

$$[J_{q^0}, h^0, \mu] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = -\nabla w^0(q^*, 0),\ A_{12} = [0\ E]$$

$$A_{21} = [0\ E^T]^T$$

$$A_{22} = \begin{bmatrix} \nabla C^0(h^*, 0) & -\Lambda^{0T} \\ \Lambda^0 & 0 \end{bmatrix}$$

then

$$[J_{q^0}, h^0, \mu]^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$B_{11} = A_{11}^{-1} + (A_{11}^{-1} A_{12})\sigma^{-1} (A_{21} A_{11}^{-1}) = A_{11}^{-1}$$

$$B_{21} = -\sigma^{-1} (A_{21} A_{11}^{-1})$$

$$B_{12} = -(A_{11} A_{21}^{-1})\sigma^{-1},\ B_{22} = \sigma^{-1}$$

$$\sigma = A_{22} - A_{21} (A_{11}^{-1} A_{12})$$

Once the derivatives of decision variables with respect to the perturbation parameter, one can obtain the derivatives of objective function at the upper level can be calculated as follows:

$$\nabla\tilde{\Pi}(p) = \nabla_p\Pi(p,h) + [\nabla_q\Pi(p,h), \nabla_h\Pi(p,h), 0] \begin{bmatrix} \nabla_\varepsilon q \\ \nabla_\varepsilon h \\ \nabla_\varepsilon \mu \end{bmatrix} \quad (32)$$

$$= \nabla_p\Pi(p,h) + \nabla_h\Pi(p,h)\ \nabla_\varepsilon h$$

$$\nabla\tilde{g}(p) = \nabla_p g(p,h) + [\nabla_q g(p,h), \nabla_h g(p,h), 0] \begin{bmatrix} \nabla_\varepsilon q \\ \nabla_\varepsilon h \\ \nabla_\varepsilon \mu \end{bmatrix} \quad (33)$$

$$= \nabla_p g(p,h) + \nabla_h g(p,h)\ \nabla_\varepsilon h$$

Algorithm

Since we have obtained the derivatives of the objective and constraint functions, we can now start to develop the algorithm for solving the reduced forms of RPEP (25). We adopt the feasible direction method originally developed by Zoutendijk (1960). In the following steps, we assume that the maximization problem is reformulated as the minimization problem with the objective function $\tilde{\Pi}$ (p, h) and constraint g (p, h) ≤ 0.

Step 1 Choose the initial feasible point $p \in R^{Ixk2}$ such that the corresponding lower level solution $h^1(p^1)$ satisfy the break-even constraint.

Step 2 Calculate

$$\tilde{\Pi}(p^1) = \Pi\left(p^1, h^1\right)$$

and set k=1.

Step 3 Find the gradients of the objective and constraint functions, $\nabla\tilde{\Pi}(p^k), \nabla\tilde{g}(p^k)$, together with $\tilde{g}(p^1) = g\left(p^k, h^k\right)$.

Step 4 Solve the following problem:

$$\text{min.:} \quad \eta$$

$$\text{s.t.} \quad \nabla\tilde{\Pi}(p)y - \eta \le 0$$

$$\tilde{g}(p) = \nabla\tilde{g}(p)\, y - \eta s \le 0$$

$$\sum_i |y_i| \le 1$$

$$\text{where} \quad s = (1, 1, ...,1)$$

Step 5 If the optimal solution h* satisfies $h^* > \omega_1$ for a predetermined positive constant ω_1, then the algorithm stops. Otherwise, go to the next step.

Step 6 Find an optimal step size α^k by solving

$$\min{}_{\alpha}\left[\tilde{\Pi}\left(p^k + \alpha y^k\right) \mid g\left(p^k + \alpha y^k\right) \le 0\right]$$

and provide the next feasible point as

$$p^{k+1} = p^k + \alpha^k y^k$$

Step 7 Solve the lower problem, given p^{k+1}, and find the solution h^{k+1} and the corresponding value of the objective function

$$\tilde{\Pi}(p^{k+1}) = \Pi\left(p^{k+1}, h^{k+1}\right)$$

Step 8 If for predetermined positive constants, ω_2 and ω_3,

$$\tilde{\Pi}(p^k) - \tilde{\Pi}(p^{k+1}) < \omega_2 \text{ and } p^{k+1} = p^k < \omega_3,$$

then the algorithm terminates. Otherwise, set k=k+1 and return to Step 3.

SUMMARY

The conventional applications of the Ramsey rule to pricing of transportation services which have been discussed by economists are unrealistic and insufficient in a sense that network congestion effects on pricing have been neglected. In this paper we proposed a Ramsey price equilibrium model in which the Ramsey price rule is restructured within the framework of multimodal network equilibrium. We showed that if network congestion effects are taken into account, Ramsey price equilibrium model can be formulated as a bilevel programming problem and that if we can replace the follower-program with its necessary and sufficient conditions, Ramsey price equilibrium can be transformed to the usual nonlinear programming problem so that many standard constrained optimization techniques can be applied. In this paper we adopt the feasible direction method and proposed an algorithm which combines the feasible direction method with the nonlinear sensitivity analysis.

It should be noted that the break-even constraint depends on the amount of government subsidy or fixed costs and that the lower the government subsidy becomes the greater the second-best prices are. This implies that the Ramsey pricing rule may result in prohibitive prices in some cases which may have income effects. What is the desirable prices level can not be determined within the framework of the Ramsey pricing rule discussed here. However, if the upper level of prices are given in a significant way, the Ramsey equilibrium price approach may be possible to show the minimum level of the government subsidy to achieve welfare maximization.

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