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A SENSITIVITY ANALYSIS BASED ALGORITHM FOR THE CONGESTED ORIGIN-DESTINATION MATRIX ESTIMATION PROBLEM

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Abstract

This paper presents a bi-level programming formulation of the congested origin-destination (O-D) matrix estimation problem. The conventional generalised least squares method is combined with an equilibrium assignment model formulated as variational inequalities. A sensitivity analysis based algorithm is developed and illustrated with two numerical examples.

INTRODUCTION

Estimation of origin-destination (O-D) matrices from traffic counts has been the subject of many research studies. Conventional methods for O-D matrix estimation assume that the route choice proportions between each O-D pair are determined independently from the estimation process. Namely, the users' route choices are considered to be independent of the O-D travel demand. This assumption of independence has inherent shortcomings (Yang et al. 1992; Yang 1995b). Because O-D matrix is estimated from observed link flows with fixed route choice proportions, and the O-D matrix is, in general, assigned to the network with user-equilibrium, there is an inconsistency in using one set of route choice proportions to obtain an O-D matrix from link flows, and another to obtain the link flow distribution by assigning the O-D matrix to the network. In a network with realistic congestion levels, this shortcoming becomes more apparent.

To overcome the aforementioned deficiency, it is necessary to combine the O-D matrix estimation and the network equilibrium assignment into one process so that the effects of traffic congestion on travel times and hence on route choices are taken into account explicitly. This attempt has been made recently by some authors. Oh (1992) examined the simultaneous estimation of O-D matrices and proposed three different solution methods: penalty function method, extrapolation method and perturbation method. Florian and Chen (1995) presented a bi-level programming formulation of the congested O-D matrix estimation problem, and developed a coordinate descent solution method.

Yang et al. (1992, 1994) and Yang (1995b) have shown that the bi-level programming approach can be used as an efficient technique to achieve the simultaneous estimation of O-D matrix and route choice under congested traffic condition. The previous generalized least squares estimation model has been combined with an equilibrium traffic assignment in the form of a bi-level optimization problem. The upper-level problem seeks to minimize the sum of error measurements in traffic counts and O-D matrix, while the lower-level problem represents a network equilibrium assignment which guarantees that the estimated O-D matrix and corresponding link flows satisfy the user-equilibrium conditions. This bi-level model has the advantage that it presumes equilibrium assignment but does not require counts for all links nor does it require counts to be error-free.

This paper describes some recent developments conducted by the author in this subject (Yang et al. 1992, 1994; Yang 1995b). The conventional generalized least squares model is integrated with equilibrium assignment which is formulated as variational inequalities. A heuristic solution algorithm is presented, in which sensitivity analysis method is used to calculate the derivatives of link flows with respect to O-D demand. Two numerical examples are given to illustrate the bi-level estimation model and the sensitivity analysis based algorithm.

THE BI-LEVEL O-D MATRIX ESTIMATION MODEL

Congestion effects in the O-D matrix estimation model are considered by recognizing the dependence of travel cost on the flow. For generality, the link cost function is assumed to depend on the flows of all links in the network:

$$c_a(\mathbf{v}) = c_a(v_1, v_2, \ldots, v_L), \ a \in A \tag{1}$$

where $\mathbf{v}=(v_1,v_2,\ldots,v_L)^T$ is the vector of network link flows, L is the number of network links and A is their set. The mathematical problem describing the user-optimal route choices can be formulated as a variational inequality:

$$\mathbf{c}(\mathbf{v})^T \cdot (\mathbf{e} - \mathbf{v}) \geq 0, \quad \text{for all } \mathbf{e} \in \Omega(\mathbf{t}) \tag{2}$$

where $\mathbf{c}(\mathbf{v})=(c_1,c_2,...,c_L)^T$ is the vector of network link costs, $\mathbf{t}=(t_1,t_2,...,t_N)^T$ is the O-D matrix rearranged as a vector, N is the number of O-D pairs, $\Omega(\mathbf{t})$ is the set of feasible link flow solutions associated with O-D matrix $\mathbf{t}$, and $\mathbf{v}$ is the user-optimal link flow solution.

Here, the generalized least squares (GLS) estimation method is extended to the congested network case with uncertain input data. A major attraction of GLS is that it allows the combination of survey data relating directly to O-D flows with traffic count data, while taking into account the relative accuracy of these data sources (Bell 1990).

The bi-level optimization model thus has the following form:

$$\min F(\mathbf{t}) = (\bar{\mathbf{t}}-\mathbf{t})^T \mathbf{U}^{-1}(\bar{\mathbf{t}}-\mathbf{t}) + (\bar{\mathbf{v}}-\mathbf{v})^T \mathbf{V}^{-1}(\bar{\mathbf{v}}-\mathbf{v}) \tag{3}$$

subject to

$$\mathbf{t} \geq 0 \tag{4}$$

where $\mathbf{v}(\mathbf{t})$ solves

$$\mathbf{c}(\mathbf{v})^T \cdot (\mathbf{e}-\mathbf{v}) \geq 0, \quad \text{for all } \mathbf{e} \in \Omega(\mathbf{t}) \tag{5}$$

where

$\mathbf{U}$, $\mathbf{V}$ are weighting matrices, $\bar{\mathrm{t}} = (\bar{\mathrm{t}}_1, \bar{\mathrm{t}}_2, \cdots \bar{\mathrm{t}}_N)^T$ is a vector representing a target O-D matrix, $\bar{\mathrm{v}} = (\bar{\mathrm{v}}_1, \bar{\mathrm{v}}_2, \cdots \bar{\mathrm{v}}_L)^T$ is a vector representing observed link flows.

The bi-level problem (3)-(5) is to estimate an O-D matrix, $\mathbf{t}$, based on a target O-D matrix, $\bar{\mathrm{t}}$ and a set of observed link flows, $\bar{\mathrm{v}}$, subject to the estimated O-D matrix, $\mathbf{t}$, and corresponding link flows, $\mathbf{v}$, satisfying a user-equilibrium condition. Note that in the above formulation we have assumed $\mathbf{v}$ and $\bar{\mathrm{v}}$ have the same dimension, or in other words, flow observations on all links are assumed to be available. The bi-level model is, however, also applicable to the situation where only a subset of links on the network are observed. In this case, the distance measurement in traffic counts (the second term of the right hand side in (3)) must be defined only for the links with observed flows.

The formulation of O-D matrix estimation problem (3)-(5) takes the generic form of a bi-level optimization problem. This bi-level formulation makes the estimation models advantageous in three aspects: 1) The model always has feasible solutions. The feasibility of the model is guaranteed even if the traffic counts are not consistent. 2) The model will work with only a subset of observed link flows. 3) The route choice proportions are determined endogenously, and the consistent equilibrium link flows and O-D matrix are determined simultaneously.

A HEURISTIC ALGORITHM

Bi-level programming problems are generally difficult to solve because evaluation of the upper-level objective function requires solution of the lower-level optimization problem. Furthermore, since the lower-level problem is in effect a non-linear constraint, the problem is a non-convex programming problem. Non-convexity portends the existence of local solutions, and thus a global optimum might be difficult to find (Friesz et al. 1990; Yang and Yagar 1994, 1995; Yang et al. 1994).

However, since the upper-level objective function of the bi-level estimation problem (3)-(5) is strongly convex with respect to both variables $\mathbf{t}$ and $\mathbf{v}$, a local minimum is likely to be a global minimum. This could be particularly likely when the target O-D matrix, used as an initial solution, is close to the true one.

Here we assume that the Jacobian matrix of the link cost function $\mathbf{c}(\mathbf{v})=(c_1(\mathbf{v}), c_2(\mathbf{v}),...,c_L(\mathbf{v}))^T$ is strictly positive definite so that there is a unique equilibrium flow distribution, $\mathbf{v}(\mathbf{t})$, obtained from the lower-level equilibrium assignment problem for any given O-D matrix $\mathbf{t}$. $\mathbf{v}(\mathbf{t})$ is also called the response or reaction function. A successful solution of O-D matrix will greatly depend on how to evaluate the reaction or response function, **v**(**t**), or in other words, how to predict link flow changes in response to O-D matrix adjustment. Since **v**(**t**) is nonlinear and its functional form is not explicitly known, prediction of variations in link flows cannot be carried out explicitly. The approach to avoiding this difficulty is to formulate a local linear approximation of **v**(**t**) at the current solution point (**t***,**v**(**t***)), based on a set of derivatives, **Q**=[q_{aw}] where q_{aw} is calculated as:

$$q_{aw} = \frac{\partial v_a}{\partial t_w}, \quad a \in A, \ w \in W. \tag{6}$$

The derivatives are used to predict direction and magnitude of changes in link flow pattern in response to changes in O-D matrix.

The general scheme of the sensitivity analysis based heuristic algorithm has the form:

Step 0. Determine an initial solution matrix $\mathbf{t}^{(0)}$; set k=0.

Step 1. Solve the lower-level equilibrium assignment problem for given $\mathbf{t}^{(k)}$.

Step 2. Calculate the derivative $\mathbf{Q}^{(k)}$ based on the sensitivity analysis method.

Step 3. Solve the upper-level estimation problem to obtain $\mathbf{t}^{(k+1)}$ using $\mathbf{Q}^{(k)}$.

Step 4. If stopping criterion is met, then stop; else let k:=k+1 and go to Step 1.

Each step of the procedure is explained below.

At *Step 0*, the target O-D matrix, $\bar{\mathbf{t}}$, might be used as an initial matrix $\mathbf{t}^{(0)}$.

At *Step 1*, the lower-level network equilibrium problem with link flow interactions is solved using iterative diagonalization procedure where the standard user-equilibrium sub-problem at each iteration is solved by the convex combination method (Sheffi 1985). Note that, in order to calculate the derivatives, the used routes and distribution of demand among these routes for each O-D pair are required to be saved from iteration to iteration. In the final output, the iterative diagonalization procedure generates information necessary for implementation of sensitivity analysis, including a complete set of equilibrium link flow pattern, a set of minimum time routes used by the drivers between each O-D pair.

At *Step* 2, the derivative information is obtained by implementing sensitivity analysis for a given solution of the network equilibrium problem. The sensitivity analysis method for equilibrium network flow has been originally developed by Tobin and Friesz (1988), and extended and applied by Yang (1995a), Yang and Yagar et al. (1994), Yang and Yagar (1994, 1995) in the development of the combined traffic assignment and traffic control models.

At *Step 3*, based on the derivative information obtained at *Step 2*, the implicit, nonlinear reaction function, **v**=**v**(**t**), is linearly approximated as:

$$\mathbf{v}(\mathbf{t}) \approx \mathbf{v}(\mathbf{t}^*) + \mathbf{Q}(\mathbf{t} - \mathbf{t}^*) \tag{7}$$

where (**t***,**v**(**t***)) is the current solution. Then the upper-level problem is approximated as a quadratic programming problem, to which the solution can be easily obtained using conventional standard method. Especially when the non-negative constraint, , is omitted, the matrix solution can be obtained explicitly as:

$$\mathbf{t}^{(k+1)} = (\mathbf{U}^{-1} + \mathbf{Q}^{(k)T}\mathbf{V}^{-1}\mathbf{Q}^{(k)})^{-1}(\mathbf{U}^{-1}\bar{\mathbf{t}} + \mathbf{Q}^{(k)T}\mathbf{V}^{-1}(\bar{\mathbf{v}} - \mathbf{v}^{(k)} + \mathbf{Q}^{(k)}\mathbf{t}^{(k)})) \tag{8}$$

At *Step 4*, the relative change rate between successive solutions of O-D matrix, $\mathbf{t}^{(k+1)}$ and $\mathbf{t}^{(k)}$, could be chosen as the stopping criterion. Namely, if

$$\max_w \left| t_w^{(k+1)} - t_w^k \right| / t_w^k \le \varepsilon \tag{9}$$

then stop, where ε is a predetermined tolerance.

NUMERICAL EXAMPLES

Example 1. The first example is concerned with O-D matrix estimation problem with a small network. The network is shown in Figure 1 with 2 O-D pairs (1→6, 2→5) and 8 links (links 1~8). The following quadratic cost functions with flow interaction are assumed:

The target O-D matrix and observed link flows (links 1, 2, 3, 4 are assumed to be observed) are given below, respectively.

$$\bar{t}_{16} = 50,\ \bar{t}_{25} = 50,$$
$$\bar{v}_1 = 20,\ \bar{v}_2 = 30,\ \bar{v}_3 = 25,\ \bar{v}_4 = 30.$$

For convenience, the weighting matrices, **U** and **V** in the estimation model (3)-(5) are assumed to be identity matrices of dimension (2x2) and (4x4), respectively.

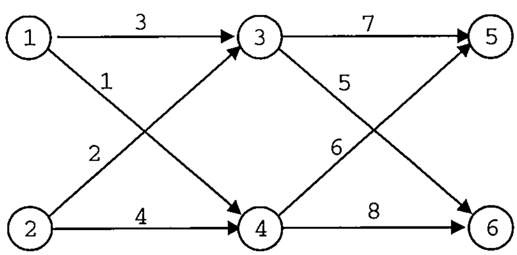

Figure 1 Network used in Example 1

$$c_1 = 6+\frac{v_1^2}{100}+\frac{v_4^2}{200},\quad c_2 = 5+\frac{v_2^2}{100}+\frac{v_3^2}{200},\quad c_3 = 8+\frac{v_3^2}{100}+\frac{v_2^2}{200},\quad c_4 = 7+\frac{v_4^2}{100}+\frac{v_1^2}{200},$$

$$c_5 = 8+\frac{v_5^2}{200}+\frac{v_8^2}{300},\quad c_6 = 6+\frac{v_6^2}{200}+\frac{v_7^2}{300},\quad c_7 = 7+\frac{v_7^2}{200}+\frac{v_6^2}{300},\quad c_8 = 8+\frac{v_8^2}{200}+\frac{v_5^2}{300}.$$

Table 1 Numerical results for Example 1

k	t_{16}	t_{25}	F(t)
0	50.00	50.00	247.89
1	46.34	51.60	218.73
2	46.20	51.63	219.51
3	46.19	51.63	218.79
Optimal solutions: t_{16}^*=46.6, t_{25}^*=52.1, F^*=209.51			

Note: stopping tolerance $\varepsilon=1.0\times10^{-3}$

Table 1 indicates the solution results for each iteration of the algorithm from an initial solution of target matrix. It can be seen that the algorithm has a fast convergence and converges to the vicinity of the global optimum (46.6, 52.1). Here the global optimum has been obtained by exhaustive enumeration of all the combinations of O-D matrices in the range, $40 \le t_{16} \le 60$, $40 \le t_{25} \le 60$, with increment size δt=0.1. However, it should be pointed out that the algorithm cannot ensure an improved solution at each step in terms of the objective value reduction. Also, it cannot be guaranteed that the converged point is always a global optimum because of the inherent non-convexity of the bi-level programming problems.

Example 2. This example presents the computational results using a larger network. The network as depicted in Figure 2 contains 9 nodes and 24 links. Four O-D pairs (1→9, 3→7, 7→3, 9→1) are assumed with a target O-D matrix:

$$\bar{t}_{19} = 60,\ \bar{t}_{37} = 80,\ \bar{t}_{73} = 70,\ \bar{t}_{91} = 60.$$

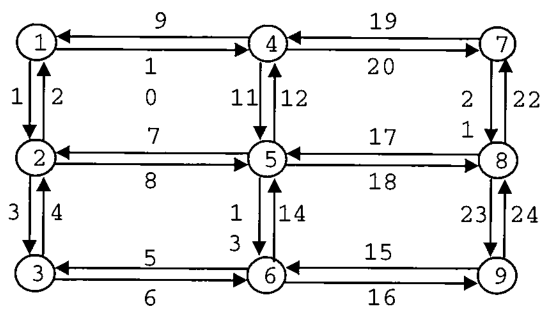

Figure 2 Network used in Example 2

Table 2 Input data for Example 2

Link	c_a^0	z_a	Link	c_a^0	z_a
(1,2)	3.0	50	(5,6)	3.0	60
(2,1)	3.0	50	(6,5)	3.0	60
(2,3)	4.0	50	(9,6)	9.0	50
(3,2)	4.0	50	(6,9)	9.0	50
(6,3)	8.0	60	(8,5)	8.0	70
(3,6)	8.0	60	(5,8)	8.0	70
(5,2)	7.0	70	(7,4)	7.0	60
(2,5)	7.0	70	(4,7)	7.0	60
(4,1)	8.0	60	(7,8)	4.0	50
(1,4)	8.0	60	(8,7)	4.0	50
(4,5)	3.0	60	(8,9)	3.0	50
(5,4)	3.0	60	(9,8)	3.0	50

Table 3 Numerical results for Example 2

k	t_{19}	t_{37}	t_{73}	t_{91}	F(t)
0	60.00	80.00	70.00	60.00	1.7570×10^3
1	73.27	89.75	75.74	67.20	0.6946×10^3
2	73.54	87.67	73.45	67.89	0.6796×10^3
3	73.54	87.88	73.59	68.23	0.6346×10^3

Note: stopping tolerance $\varepsilon=5.0\times10^{-3}$

In this example, link travel time is assumed to be dependent only on the flow on that link (without flow interaction). The standard BPR (Bureau of Public Road) formula is adopted as the congested link cost function:

$$c_a(v_a) = c_a^0\left\{1.0 + 0.15\left(\frac{v_a}{z_a}\right)^4\right\} \tag{10}$$

where z_a and c_a^0 are the capacity and free-flow travel time of link $a \in A$, respectively. Table 2 contains the link cost function information (c_a^0, z_a). The observed links are assumed to be (5~10) with observed flows:

$$\bar{v}_5 = 40,\ \bar{v}_6 = 50, \bar{v}_7 = 65,\ \bar{v}_8 = 70,\ \bar{v}_9 = 35,\ \bar{v}_{10} = 50.$$

Table 3 provides the changes of the estimated O-D matrices, upper-level objective values with respect to major outer iterations. From this table, it can be seen that the algorithm has a fast convergence even for medium-sized network. The sensitivity analysis based algorithm might be considered to be efficient in solving the congested O-D matrix estimation problem.

CONCLUSIONS

This paper presented the bi-level congested O-D matrix estimation model and the sensitivity analysis based algorithm. The estimation model is formulated as a bi-level programming problem with a variational inequality constraint. The numerical examples showed that the algorithm has fast convergence and might be efficient in congested O-D matrix estimation problems for large networks. Further work should be conducted on the statistical characteristics of the estimates obtained by the bi-level estimation model.

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