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TRAFFIC FLOW IN DENSE NETWORKS

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Abstract

This paper considers a new approach for modelling the traffic flow in dense networks by using the boundary element technique (B.E.M), which needs information data from a very limited number of points from the network. The purpose of this communication is to provide a mathematical foundation for the study of this approach.

INTRODUCTION

Most cities in the world share a common problem: urban traffic congestion. To analyze and quantify this problem several techniques have been available during the last decade: from simple mathematical models to the most thorough modelling techniques using highly detailed network systems subjected to dynamic assignment under random data. This paper considers a new approach for modelling the traffic flow in dense networks by using a sophisticated mathematical continuum model, which needs (as input) information data from a very limited number of points from the network. The purpose of this communication is to provide a mathematical foundation for the study of this approach.

We postulate that this methodology can speed up the computation of relevant values concerning the time evolution of traffic on a city network. The technique is very promising and it complements the traditional procedures based on the modelling of a city by networks made of nodes and links.

The modelling of traffic systems is, at present, mainly based on discretizing the region to be analyzed by a discrete network. This approach might be valid and convenient for the study of not very extensive regions or/and not very road-dense zones. In case the real road and street network are complex (the number of roads is large enough, the density of roads impedes a proper and reliable modelling, the origins and destinations are difficult to be precisely pointed out, etc.) the network theory method would not be an adequate tool. The errors made in the modelling of the network itself (roads not considered, missing interconnection, crossings not well defined, etc.) will introduce a savage uncertainty on the results obtained by solving problems based on this model. Moreover, the more precise the model is defined the longer its size becomes, making the algorithms used for the solution impractical from a computational point of view. In these cases it might be more convenient to model the region to be studied under a continuum representation. In the last 25 years several researchers have proposed different methods for dealing with the traffic distribution in dense urban regions. We should acknowledge the works by Vaughan (1987) and Newell (1980). Most of these methods were only applied to simple geometries that are far away from a practical use, although they defined the foundation for further research.

The purpose here is to develop a practical method to solve problems of continuum modelling in traffic networks ready to be implemented for obtaining the solution of practical scenarios.

The basic of our model is presented in the next section, where the constitutive equations of a continuum model are described. The following section constitutes the core of the technique proposed. The last two sections indicate the way a real area with multicommodity flow should be dealt with.

BACKGROUND

Basic concepts on traffic flow

The first investigators who formulated the equation for traffic flow, from a macroscopic continuum modelling point of view were Lighthill et al. (1955). The only equation formulated in this model is that of continuity in a monodimensional space:

$$\frac{\partial K}{\partial t} + \frac{\partial Q}{\partial x} = \Theta(x,t) \qquad (1)$$

which expresses the balance between the number of vehicle's growth rate (traffic density or concentration K) in a road section, the net flow (Q=K V) across the section and a generation function (Θ). In this model, the *Simple Continuum Model*, the mean space speed is supposed to be dependent on traffic density $V = V_e(K)$.

This model is clearly a very simplistic modelling of traffic flow that does not allow for speed fluctuations around equilibrium values, since it supposes that the speed depends on the density, and which allows for appearing shock waves, implying instantaneous speed changes.

A later modelling, the *Payne model* (1971) has been by far the most used among traffic scientist. This is a second order model that allows for fluctuations in the vehicle speed by introducing this variable (V) as an unknown into the model. In order to do that the momentum equation is needed:

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = \frac{V_e - V}{T} - \frac{\nu}{T}\frac{1}{K}\frac{\partial K}{\partial x} \tag{2}$$

where T is the driver's reaction time, ν is an anticipation coefficient, V_e the equilibrium speed and x and t indicate space and time respectively.

It has been proved, Del Castillo et al. (1994) that "the capabilities of the Payne model to solve compression shock waves in traffic flow in a continuous manner is limited" and that "the results of the Payne model may be fairly well approximated by those of the *Simple Continuum Model*". This assertion takes place for the cases in which the characteristic variation time of the traffic flow conditions is greater than the reaction time and, then, the relaxation term in (2) may be neglected, and the Payne model simplifies to the *Simple Continuum Model.* As stated in Del Castillo (1994) "the last condition is likely to be met in those situations where the changes of the traffic state take place slowly in comparison with the time required by the drivers to reach their desired speed. But those situations are the common situations, because the reaction time of drivers is small, namely about 0.6 sec. Thus, the differences between the solutions of the revised Payne model and those of the simple continuum model should be irrelevant in all realistic situations". Under the former hypotheses the adoption of the *Simple Continuum Model* (1), with a specified volume-speed-density relationship, is not a bad choice.

Since 1925, with the work of Schaar (1925), several volume-speed-density relationships have been proposed, although the most used are those authored to Greenshilds (1935), Greenberg (1959), Underwood (1961), Drake et al. (1967) and Pipers (1967) (see reference Del Castillo et al. (1995a)). The acceptance, by the scientific community, of a general analytical expression for the speed-density relationship is still an open problem. The properties (Del Castillo et al. (1995a, 1995b)) that this expression should hold impose such severe restrictions that the finding of such curves it is not an easy task (the formerly named relationships do not satisfy at least a couple of needed properties).

An attempt to draw some light on this open problem is the paper by Del Castillo et al. (1995b), in this work a functional form for the speed-density relationship is presented which includes several examples of generating functions contrasted with real experimental data. In that investigation the authors claim that the parameters that characterize the equilibrium traffic flow relationship are the *jam density* (K_j), the *kinematic wave speed at jam density* (C_j), both of them considered as traffic flow constants and the *free flow speed* (V_f) which varies over a much larger range as it is strongly dependent on road (street) characteristics.

From this work a family of speed-density expressions has been proposed:

$$u = u_f[1 - F(\vartheta)], \quad \vartheta = \frac{1}{u_f}(\frac{1}{k} - 1) \tag{3}$$

where

$$u = \frac{V}{|C_j|}, k = \frac{K}{K_j}, C_j = \left.\frac{\partial Q}{\partial K}\right|_{Kj}, \tag{4}$$

and $F(\vartheta)$ is the *generating* function. Some expressions for the *generating* function are published in Del Castillo et al. (1995b).

From (1), (2) and (3) and the non-dimensional time $\tau \doteq t|C_j|K_j$ the following model is defined:

$$\frac{\partial k(x,\tau)}{\partial \tau} + \frac{\partial q(x,\tau)}{\partial x} = \frac{1}{K_j |C_j|}\Theta(x,\tau)$$
$$q(x,\tau) = k(x,\tau)u(x,\tau) = u_f(1 - F(\vartheta)) \quad (5)$$

The above expressions can be extended to two-dimensional regions as it is postulated in the next section.

Basic concepts on two dimensional traffic flow

The basic concepts are based in the works of Dafermos (1980) and Newell (1993) on dense urban street grids. In traffic theory we named *dense grids* to a grid of roads and streets packed enough to assume that the spacing between them and the length of any one of them are small compared to the whole region analyzed. Such a grid can be approximated by a continuous space with appropriate characteristics. On such a continuous space the orientation of roads will depend on the zone being observed. Thus, if a zone has roads in almost any orientation, this zone will be considered an isotropic- road area, on the other hand if a zone has roads orientated in almost only two directions this is to be assumed an orthotropic-road zone. In general, any spatial region can be divided, according to the orientation of the roads and streets, into different zones, with isotropic, orthotropic or anisotropic geometric characteristics.

Let Ω denote the under considered urban street grid that occupies a region in E_2; where E_2 stands for the two-dimensional Euclidean space. Let x = (x, y) be the position vector of a point in E_2. In an isotropic-road zone Ω the traffic flow of a single-commodity, only one pair O/D, would satisfy the conservation equation:

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} + \nabla \cdot f(\mathbf{x},t) = \rho(\mathbf{x},t), \quad (6)$$

where $f(\mathbf{x},t)$ stands for the traffic flow (vehicles per unit of time and unit of length) vectorial function (fx, fy), $\rho(\mathbf{x},t)$ the generating function ($\rho(\mathbf{x},t)$ will be positive if flow is originated at x or negative when it is absorbed) and $\phi(\mathbf{x},t)$ the traffic density or concentration (per unit of surface area).

By presuming that the traffic flow is not only function of velocity and density but function also of density gradient, one can write:

$$\begin{aligned} f(\mathbf{x},t) &= -\mu\nabla\phi(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\phi(\mathbf{x},t) \\ \mathbf{u}(\mathbf{x},\tau) &= [ux(\mathbf{x},\tau), uy(\mathbf{x},\tau)], \\ ux(\mathbf{x},\tau) &= uf\,\phi(\mathbf{x},\tau)(1 - \mathrm{F}x(\vartheta)), \\ uy(\mathbf{x},\tau) &= uf\,\phi(\mathbf{x},\tau)(1 - \mathrm{F}y(\vartheta)). \end{aligned} \quad (7)$$

Variational formulation

If we assume that a vehicle moving from point x_1 to x_2 incurs a travel cost that depends upon the traffic flow, we are taking into account the network geometric characteristics and the congestion effect:

$$\bar{\mathbf{c}} = \hat{\mathbf{c}}(f,\mathbf{x}) \quad (8)$$

Under this assumption, a user vehicle traveling along a path l will incurs a cost given by:

$$c_l = \int_l \bar{\mathbf{c}}(f,\mathbf{x})dl = \int_0^{fl} \mathbf{c}(f,\mathbf{x})df, \quad (9)$$

where $\mathbf{c}(f,\mathbf{x})$ is the travel cost used in the formulation of equilibrium assignment problem. Extending the travel cost $\mathbf{c}_l$ to all possible paths inside region Ω the total uni-commodity travel cost for that region will be formulated as (Newell (1993)):

$$T(f) = \int_{\Omega}\left[\int_{0}^{f} \mathbf{c}(f,\mathbf{x})\,df\right] dA \tag{10}$$

The solution of the assignment problem sought will be obtained minimizing (10) subjected to restrictions (6), (7) and boundary conditions on Ω, where Ω represents the region analyzed, A its surface, $\mathbf{c}(f,\mathbf{x})$ the cost function of a trip which depends on the traffic flow level in the surrounding and f the traffic flow level itself.

The cost function $\mathbf{c}(f,\mathbf{x})$ will depend on the location point x into the region Ω and on the flow traversing that point $f(\mathbf{x},t)$ in such a way that it can be expressed as follows:

$$\mathbf{c}(f,\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{b}|f(\mathbf{x})|, \tag{11}$$

where the flow function appears as a non-negative term since $\mathbf{c}(f,\mathbf{x})$ should always be a non-negative function.

FORMULATION OF ONE-COMMODITY TWO-DIMENSIONAL FLOW

Substituting (7) into (6), the following transient equation is obtained:

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t} - \mu\nabla^2\phi(\mathbf{x},t) + \nabla[\mathbf{u}(\mathbf{x},t)\phi(\mathbf{x},t)] = \rho(\mathbf{x},t). \tag{12}$$

As the velocity u varies in space, assuming that this variation ($\tilde{\mathbf{u}}$) is small in the neighbourhood of an equilibrium speed $\overline{\mathbf{u}}$, we can write:

$$\mathbf{u} = \overline{\mathbf{u}} + \tilde{\mathbf{u}}. \tag{13}$$

Substituting (13) into (12), the resulting expression is:

$$\mu\nabla^2\phi - \overline{\mathbf{u}}\nabla\phi = \frac{\partial\phi}{\partial t} - \rho + \nabla(\tilde{\mathbf{u}}\phi), \tag{14}$$

where for the sake of brevity the dependence on spatial variable x and time t is taken for granted.

Applying a weighted residual technique to the above equation over the whole region Ω, using as weighting function ψ the fundamental solution of the steady-state equation (see Appendix):

$$\mu\nabla^2\psi + \overline{\mathbf{u}}\nabla\psi = -\delta(\xi), \tag{15}$$

we obtain:

$$\int_{\Omega}(\mu\nabla^2\phi - \overline{\mathbf{u}}\nabla\phi)\,\psi\,d\Omega = \int_{\Omega}\left[\frac{\partial\phi}{\partial t} - \rho + \nabla(\tilde{\mathbf{u}}\phi)\right]\psi\,d\Omega. \tag{16}$$

Integrating the left hand side of the former expression by parts twice, taking into account (15) and that $\nabla\overline{\mathbf{u}} = \mathbf{0}$, it yields:

$$\phi(\xi) + \mu\int_{\Gamma}\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)d\Gamma + \int_{\Gamma}\phi\psi\overline{\mathbf{u}}_n\,d\Gamma = -\int_{\Omega}\left[\frac{\partial\phi}{\partial t} - \rho + \nabla(\tilde{\mathbf{u}}\phi)\right]\psi\,d\Omega, \tag{17}$$

where $\bar{\mathbf{u}}_n = \mathbf{u} \cdot \mathbf{n}$, being n the unit outward normal vector to Γ, being Γ the boundary of Ω.

Integral equation (17) is valid for any point within the domain ($\xi \in \Omega$). Applying the above equation to any point on the boundary (Γ), the following expression is obtained:

$$c(\xi)\phi(\xi) + \mu \int_\Gamma (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}) d\Gamma + \int_\Gamma \phi \psi \bar{\mathbf{u}}_n d\Gamma = -\int_\Omega [\frac{\partial \phi}{\partial t} - \rho + \nabla(\tilde{\mathbf{u}}\phi)] \psi d\Omega, \tag{18}$$

where the integrals are defined in the sense of Cauchy Principal Values and $c(\xi)$ are given by $\frac{\theta}{2\pi}$, being θ the internal angle at point ξ in radians. It is noteworthy that all integrals, except the one on the right hand side of (18) are boundary integrals. If the domain integral is transformed into boundary integrals, the numerical evaluation of preceding equation is facilitated. In what follows this transformation is undertaken.

Domain integrals

Domain integrals of time derivative

To obtain a boundary integral corresponding to the derivative respect to time, the following approximation is assumed:

$$\dot{\phi} = \frac{\partial \phi}{\partial t} = \sum_{k=1}^{K_1} f_k(\mathbf{x}) \alpha(t), \tag{19}$$

where the above series involves a set of known functions f_k that depend only on geometry and a set of coefficients that depends only on time. Under this assumption, the first domain integral of the right hand side of (19) becomes:

$$\int_\Omega \frac{\partial \phi}{\partial t} \psi d\Omega = \sum_{k=1}^{K_1} \alpha_k \int_\Omega f_k \psi d\Omega. \tag{20}$$

By choosing each function f_k in such a way that they are particular solutions of the equation:

$$\mu \nabla^2 \zeta_k - \bar{\mathbf{u}} \nabla \zeta_k = f_k, \tag{21}$$

expression (20) yields:

$$\int_\Omega \frac{\partial \phi}{\partial t} \psi d\Omega = \sum_{k=1}^{K_1} \alpha_k \int_\Omega (\mu \nabla^2 \zeta_k - \bar{\mathbf{u}} \nabla \zeta_k) \psi d\Omega. \tag{22}$$

Integrating by parts in a similar way to obtain (18) from (16), expression (22) gives:

$$\int_\Omega \frac{\partial \phi}{\partial t} \psi d\Omega = \sum_{k=1}^{K_1} \alpha_k [-c(\xi)\zeta_k(\xi) - \mu \int_\Gamma (\zeta_k \frac{\partial \psi}{\partial n} - \psi \frac{\partial \zeta_k}{\partial n}) d\Gamma - \int_\Gamma \zeta_k \psi \bar{\mathbf{u}}_n d\Gamma]. \tag{23}$$

Domain integrals of spatial distribution

The simplest way to compute this domain integral is by subdividing the region into a series of internal cells Ω_c. The integration over each cell is then performed by a numerical integration similar to the finite element method:

$$\int_{\Omega} \rho\psi \, d\Omega = \sum_{c=1}^{C} \int_{\Omega_c} \rho\psi \, d\Omega_c = \sum_{c=1}^{C} \sum_{i=1}^{I} \omega_i (\rho\psi)_i \Omega_c, \tag{24}$$

where C is the total number of cells describing the original domain Ω, ω_i are the Gauss integration weights, $(\rho\psi)_i$ the value of function $\rho\psi$ at integration point i on each cell, I the number of integration points on each cell and Ω_c the area of cell c.

In order to transform the domain integral into a boundary integral, the following approximation is postulated:

$$\rho(\mathbf{x}) = \sum_{k=1}^{K_2} s_k(\mathbf{x}) \beta_k, \tag{25}$$

where the functions $s_k(\mathbf{x})$ are particular solution of the equation (21), substituting ζ_k by λ_k and f_k by s_k.

In an analogous manner to the former case, this domain integral becomes:

$$\int_{\Omega} \rho \, \psi \, d\Omega = \sum_{k=1}^{K_2} \beta_k [- c(\xi) \lambda_k(\xi) - \mu \int_{\Gamma} (\lambda_k \frac{\partial \psi}{\partial n} - \psi \frac{\partial \lambda_k}{\partial n}) d\Gamma - \int_{\Gamma} \lambda_k \, \psi \, \bar{\mathbf{u}}_n d\Gamma]. \tag{26}$$

In case concentrated sources are present at certain points ε, ($\varepsilon = 1, \ldots, P$), the function ρ at these points becomes:

$$\rho(\mathbf{x}) = \rho_\varepsilon \sum_{\varepsilon=1}^{P} \delta(\varepsilon), \tag{27}$$

where ρ_ε is the magnitude of the force, $\delta(\varepsilon)$ is a Dirac delta function and the domain integral yields:

$$\int_{\Omega} \rho\psi \, d\Omega = \sum_{\varepsilon=1}^{P} \rho_\varepsilon \, \psi(\varepsilon). \tag{28}$$

Domain integrals of divergence

In this case the approximation assumed is:

$$\nabla(\tilde{\mathbf{u}}\phi) = \sum_{k=1}^{K_3} r_k(\mathbf{x}) \gamma(t), \tag{29}$$

where the functions r_k are particular solutions of the equation:

$$\mu \nabla^2 \chi_k - \bar{\mathbf{u}} \nabla \chi_k = r_k. \tag{30}$$

From (29) and (30) the domain integral becomes:

$$\int_{\Omega} \nabla(\tilde{\mathbf{u}}\phi) \, \psi \, d\Omega = \sum_{k=1}^{K_3} \gamma_k [- c(\xi) \chi_k(\xi) - \mu \int_{\Gamma} (\chi_k \frac{\partial \psi}{\partial n} - \psi \frac{\partial \chi_k}{\partial n}) d\Gamma - \int_{\Gamma} \chi_k \, \psi \, \bar{\mathbf{u}}_n d\Gamma]. \tag{31}$$

Boundary integral equation

From expressions (18), (23), (26), (28) and (31) the following integral equation is obtained:

$$
\begin{aligned}
& c(\xi)\phi(\xi)+\mu\int_{\Gamma}\phi\frac{\partial\psi}{\partial n}d\Gamma-\mu\int_{\Gamma}\psi\frac{\partial\phi}{\partial n}d\Gamma+\int_{\Gamma}\phi\psi\bar{u}_n d\Gamma = \\
& \sum_{k=1}^{K_1}\alpha_k[c(\xi)\zeta_k(\xi)+\mu\int_{\Gamma}\zeta_k\frac{\partial\psi}{\partial n}d\Gamma-\mu\int_{\Gamma}\psi\frac{\partial\zeta_k}{\partial n}d\Gamma+\int_{\Gamma}\zeta_k\psi\bar{u}_n d\Gamma] \\
& -\sum_{k=1}^{K_2}\beta_k[c(\xi)\lambda_k(\xi)+\mu\int_{\Gamma}\lambda_k\frac{\partial\psi}{\partial n}d\Gamma-\mu\int_{\Gamma}\psi\frac{\partial\lambda_k}{\partial n}d\Gamma+\int_{\Gamma}\lambda_k\psi\bar{u}_n d\Gamma]+\sum_{\varepsilon=1}^{P}\rho_\varepsilon\psi(\varepsilon) \\
& +\sum_{k=1}^{K_3}\gamma_k[c(\xi)\chi_k(\xi)+\mu\int_{\Gamma}\chi_k\frac{\partial\psi}{\partial n}d\Gamma-\mu\int_{\Gamma}\psi\frac{\partial\chi_k}{\partial n})d\Gamma+\int_{\Gamma}\chi_k\psi\bar{u}_n d\Gamma]
\end{aligned}
\qquad (32)
$$

For the numerical solution of the problem, equation (32) can be written in a discretized form, analogous to the finite element and boundary element methods, in which the integrals over the boundary are approximated by a summation of integrals over individual boundary elements, abstracted to the following simple expression:

$$
\mathbf{w}(\xi)=\int_{\Gamma}\mathbf{f}(\mathbf{x},\xi)\mathbf{g}(\mathbf{x},\xi)d\Gamma=\sum_{e=1}^{E}\int_{\Gamma_e}\mathbf{f}(\mathbf{x},\xi)\mathbf{g}(\mathbf{x},\xi)\,d\Gamma_e, \qquad (33)
$$

where E stands for the number of boundary elements on which the contour is discretized.

The variation of generic function $\mathbf{f}(\mathbf{x},\xi)$ within each element is approximated by interpolating from the values at certain element nodes (see Figure 1).

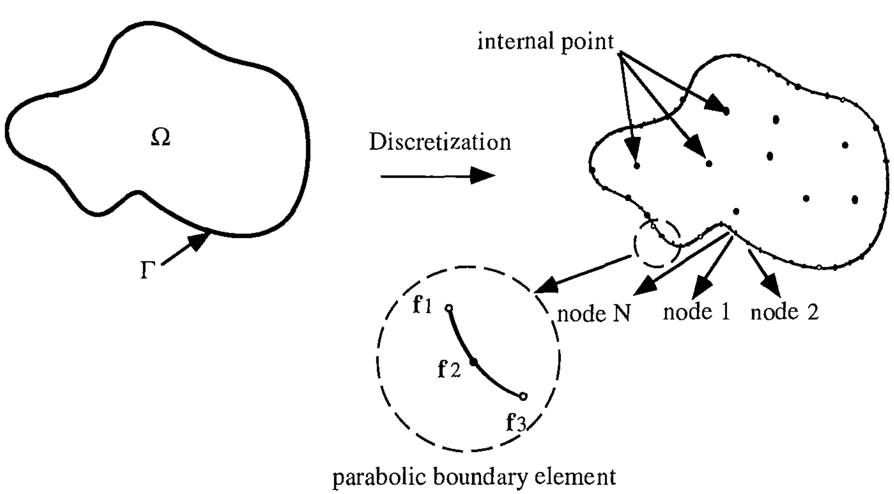

Figure 1 Boundary discretization and interior points definition

In case parabolic elements are used the following expression applies:

$$\mathbf{f}(\mathbf{x},\xi) = (\Phi_1, \Phi_2, \Phi_3) \cdot \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}, \tag{34}$$

where Φ_i are quadratic interpolation functions. Substituting (34) into (33) one gets:

$$\mathbf{w}(\xi) = \sum_{e=1}^{E} (\mathbf{a}_e^1, \mathbf{a}_e^2, \mathbf{a}_e^3) \cdot \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}_e, \tag{35}$$

where $\mathbf{a}_e^i = \int_{\Gamma e} \Phi_i \mathbf{g}(\mathbf{x},\xi) d\Gamma$.

Following the above philosophy, the left hand side of equation (32) can be discretized, for a given boundary node i (ξ) as follows:

$$c_i \phi_i + \sum_{e=1}^{E} (h_{ie}^1, h_{ie}^2, h_{ie}^3) \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_e - \sum_{e=1}^{E} (g_{ie}^1, g_{ie}^2, g_{ie}^3) \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}_e, \tag{36}$$

where scripts 1, 2, 3 identify nodal points on each element and

$$\begin{aligned} h_{ie}^j &= \mu \int_{\Gamma e} \Phi_j \left(\frac{\partial \psi}{\partial n} + \frac{\mathbf{u}_n}{\mu}\right) d\Gamma, \\ g_{ie}^j &= \mu \int_{\Gamma e} \Phi_j \, \psi \, d\Gamma. \end{aligned} \tag{37}$$

The contributions of adjoining elements to each nodal point have to be added up, getting (36) into:

$$c_i \phi_i + \sum_{e=1}^{E} H_{ie} \phi_e - \sum_{e=1}^{E} G_{ie} \left.\frac{\partial \phi}{\partial \mathbf{n}}\right|_e, \tag{38}$$

Dealing with the terms of the right hand side of expression (32), in an analogous manner ,this equation yields:

$$\begin{aligned} c_i \phi_i + \sum_{e=1}^{E} H_{ie} \phi_e - \sum_{e=1}^{E} G_{ie} \left.\frac{\partial \phi}{\partial \mathbf{n}}\right|_e = \\ \sum_{k=1}^{K1} \alpha_k \Big[c_i \psi_{ki} + \sum_{e=1}^{E} H_{ie} \psi_{ke} - \sum_{e=1}^{E} G_{ie} \left.\frac{\partial \psi_k}{\partial \mathbf{n}}\right|_e \Big] \\ - \sum_{k=1}^{K1} \beta_k \Big[c_i \lambda_{ki} + \sum_{e=1}^{E} H_{ie} \lambda_{ke} - \sum_{e=1}^{E} G_{ie} \left.\frac{\partial \lambda_k}{\partial \mathbf{n}}\right|_e \Big] + \sum_{e=1}^{E} \rho_\varepsilon \psi_i(\varepsilon) \\ + \sum_{k=1}^{K1} \gamma_k \Big[c_i \chi_{ki} + \sum_{e=1}^{E} H_{ie} \chi_{ke} - \sum_{e=1}^{E} G_{ie} \left.\frac{\partial \chi_k}{\partial \mathbf{n}}\right|_e \Big] \end{aligned} \tag{39}$$

If the boundary elements in which the contour is discretized include N nodes (see Figure 1), a collocation technique of equation (39) applied to these nodes will yield a system of N equations of the form:

$$
\begin{aligned}
\mathbf{H}\phi - \mathbf{G}\frac{\partial \phi}{\partial \mathbf{n}} = \sum_{k=1}^{K1} \alpha_k [\mathbf{H}\zeta_k - \mathbf{G}\frac{\partial \zeta_k}{\partial \mathbf{n}}] \\
- \sum_{k=1}^{K2} \beta_k [\mathbf{H}\lambda_k - \mathbf{G}\frac{\partial \lambda_k}{\partial \mathbf{n}}] + \bar{\mathbf{d}}, \\
+ \sum_{k=1}^{K3} \gamma_k [\mathbf{H}\chi_k - \mathbf{G}\frac{\partial \chi_k}{\partial \mathbf{n}}]
\end{aligned}
\tag{40}
$$

where H and G are NxN matrices and $\phi, \frac{\partial \phi}{\partial \mathbf{n}}, \zeta_k, \frac{\partial \zeta_k}{\partial \mathbf{n}}, \lambda_k, \frac{\partial \lambda_k}{\partial \mathbf{n}}, \chi_k, \frac{\partial \chi_k}{\partial \mathbf{n}}, \bar{\mathbf{d}}$ are vectors of length N. The c_i values on equation (36) have been incorporated into the diagonal coefficients of matrices H.

Matrices H and G are evaluated numerically using a Gaussian integration. The diagonal coefficients of matrices H can also evaluated through:

$$
H_{ii} = -\sum_{e=1}^{E} H_{ie}, \quad (i \neq e), \tag{41}
$$

as the solution of a system with a constant value of ϕ along the boundary can only be obtained if matrix H is singular.

Assuming that the number of terms in expansion (19) and (29) coincides with the number of boundary nodes, one can write:

$$
\dot{\phi} = \mathbf{F}\alpha, \quad \nabla(\tilde{\mathbf{u}}\phi) = \mathbf{R}\beta, \tag{42}
$$

where $\dot{\phi}, \alpha, \nabla(\tilde{\mathbf{u}}\phi), \beta$, are vectors of nodal values.

Applying (25) to N boundary nodes and M internal points (K_2=N+M) we can write $\rho = \mathbf{S}\gamma$. Interior points are defined in order to obtain a more accurate solution. These M points do not form part of any element or cell, only the coordinates are needed as input data.

Substituting (42) into (40) results in the following system of equations:

$$
\begin{aligned}
\mathbf{H}\phi - \mathbf{G}\frac{\partial \phi}{\partial \mathbf{n}} = [\mathbf{H}\zeta - \mathbf{G}\frac{\partial \zeta}{\partial \mathbf{n}}]\mathbf{F}^{-1}\dot{\phi} \\
- [\mathbf{H}\lambda - \mathbf{G}\frac{\partial \lambda}{\partial \mathbf{n}}]\mathbf{S}^{-1}\rho + \bar{\mathbf{d}} \ , \\
+ [\mathbf{H}\chi - \mathbf{G}\frac{\partial \chi}{\partial \mathbf{n}}]\mathbf{R}^{-1}\nabla(\tilde{\mathbf{u}}\phi)
\end{aligned}
\tag{43}
$$

and calling

$$
\begin{aligned}
\mathbf{B} = -[\mathbf{H}\zeta - \mathbf{G}\frac{\partial \zeta}{\partial \mathbf{n}}]\mathbf{F}^{-1} \\
\mathbf{d} = \bar{\mathbf{d}} - [\mathbf{H}\lambda - \mathbf{G}\frac{\partial \lambda}{\partial \mathbf{n}}]\mathbf{S}^{-1}\rho + [\mathbf{H}\chi - \mathbf{G}\frac{\partial \chi}{\partial \mathbf{n}}]\mathbf{R}^{-1}\nabla(\tilde{\mathbf{u}}\phi),
\end{aligned}
\tag{44}
$$

expression (43) results in:

$$
\mathbf{B}\dot{\phi} + \mathbf{H}\phi = \mathbf{G}\frac{\partial \phi}{\partial \mathbf{n}} + \mathbf{d}. \tag{45}
$$

In order to solve system (45), any standard direct time-integration scheme can be used to obtain a solution. A valid scheme would use a linear approximation for the variation of ϕ and $\mathbf{q} = \frac{\partial \phi}{\partial \mathbf{n}}$:

$$\begin{aligned} \phi &= (1-\theta_\phi)\phi^m + \theta_\phi \phi^{m+1} \\ \mathbf{q} &= (1-\theta_q)\mathbf{q}^m + \theta_\phi \mathbf{q}^{m+1}, \\ \dot{\phi} &= \frac{1}{\Delta t}(\phi^{m+1} - \phi^m) \end{aligned} \tag{46}$$

where θ_ϕ and θ_q are parameters of the approximation. Substituting (46) into (45), the following expression is obtained:

$$(\tfrac{1}{\Delta t}\mathbf{B} + \theta_\phi \mathbf{H})\phi^{m+1} - \theta_q \mathbf{G}\mathbf{q}^{m+1} = [\tfrac{1}{\Delta t}\mathbf{B} - (1-\theta_\phi)\mathbf{H}]\phi^m + (1-\theta_q)\mathbf{G}\mathbf{q}^m + \mathbf{d}. \tag{47}$$

The right hand side of the above expression is always known, since it involves values that have been calculated previously or which have been specified as initial conditions. From the boundary conditions at time $(m+1)\Delta t$, the values of ϕ are prescribed over part of the contour (Γ_ϕ) where q are unknown, along the rest of the contour (Γ_q) occurs the opposite; hence there are only N unknowns in the system (47). The time step value Δt should be small enough to ensure numerical stability. Adequate values for θ_ϕ and θ_q are 0.5 .

Introducing the boundary conditions into (47) and arranging the system by moving known terms to the right hand side, we can write:

$$\mathbf{A}\mathbf{x} = \mathbf{y}, \tag{48}$$

where x is a vector of unknown boundary values of ϕ and q, and y is a *load* vector. This system can be solved by using a standard direct procedure like Gauss elimination.

Note that the matrices H, G, R, and S depend only on geometrical data and they need to be evaluated only once.

URBAN AREAS WITH DIFFERENT ZONES

When the urban area under study is composed of different regions or zones, a subregion technique may be applied.

In most situations the area analysed involves a number of contiguous zones of different characteristics. In general, there will be a number of zones Z^l (l=1,...,L), each one enclosed by its boundary Γ^l, (see Figure 2).

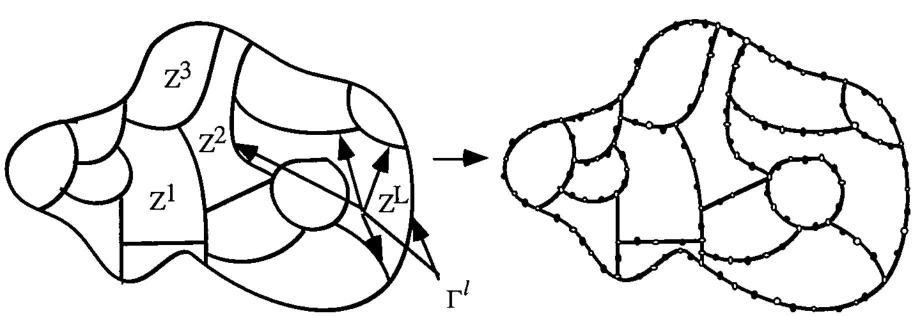

Figure 2 Urban area with different zones

The common interfaces of any pair of zones are forced to have the same traffic density ϕ , moreover the flow should ensure continuity across the shared contour.

Dealing with each zone as an entirely independent entity, an assembly process will provide a system of equations similar to (48) involving only boundary unknowns.

URBAN AREAS WITH MULTI-COMMODITY TRIPS

Expression (47) and (48) are only valid for problems with only one pair origin-destination. In case several pairs O-D are present, case that corresponds to real situations, we could write down a system of equation (given by (47) or (48)) for each pair O-D:

$$\begin{aligned} \mathbf{A}_1\mathbf{x}_1 &= \mathbf{y}_1 \quad \text{pair O–D identified as 1} \\ \mathbf{A}_2\mathbf{x}_2 &= \mathbf{y}_2 \quad \text{pair O–D identified as 2} \\ &\ldots\ldots\ldots\ldots \\ \mathbf{A}_j\mathbf{x}_j &= \mathbf{y}_j \quad \text{pair O–D identified as j}\ , \\ &\ldots\ldots\ldots\ldots \\ \mathbf{A}_J\mathbf{x}_J &= \mathbf{y}_J \quad \text{pair O–D identified as J} \end{aligned} \tag{49}$$

These correspond to an unreal uncoupled system of equations. In real situations total densities and flows are related through a similar matrix equation:

$$\mathbf{A}_t\mathbf{x}_t = \mathbf{y}_t, \tag{50}$$

where x_t and y_t are the unknown and known vectors respectively, both of them involving total densities and total flows.

Solution of (49) and (50) will assign values to the boundary unknowns. Once this step is taken, we can continue further evaluating densities and speeds for selected internal point by using expression (17).

Although we have not presented the formulation for orthotropic zones, the extension of the above formulation to this case is almost immediate.

URBAN AREAS WITH A MAIN ROAD NETWORK

During the process of modelling urban areas it is always frequent to find out that there exists , at least , tow kind of roads: main roads and secondary roads. With the technique presented on this paper one can easily model the secondary road network but not the primary (as long as it is not dense enough), but we can in all cases model the main road network as a discretized network (under a classical point of view) connected to the continuum model through specific points (source-sink points). These points will be considered as concentrated sources in the continuum model. The inclusion of the mathematical model of the discrete main network coupled to the system (49) allows the solution of the whole real problem. A scheme of this procedure is depicted in Figure 3.

CONCLUSIONS

On the basis of the model that have been developed in (6) and (7) a completely new approach based on the technique of Dual Reciprocity Boundary Element Method (Partridge et al. (1992)) for the analysis of two-dimensional traffic flow has been presented. Although this model contains several hypotheses that might be under a ban it outlines a new horizon for dealing with dense grids.

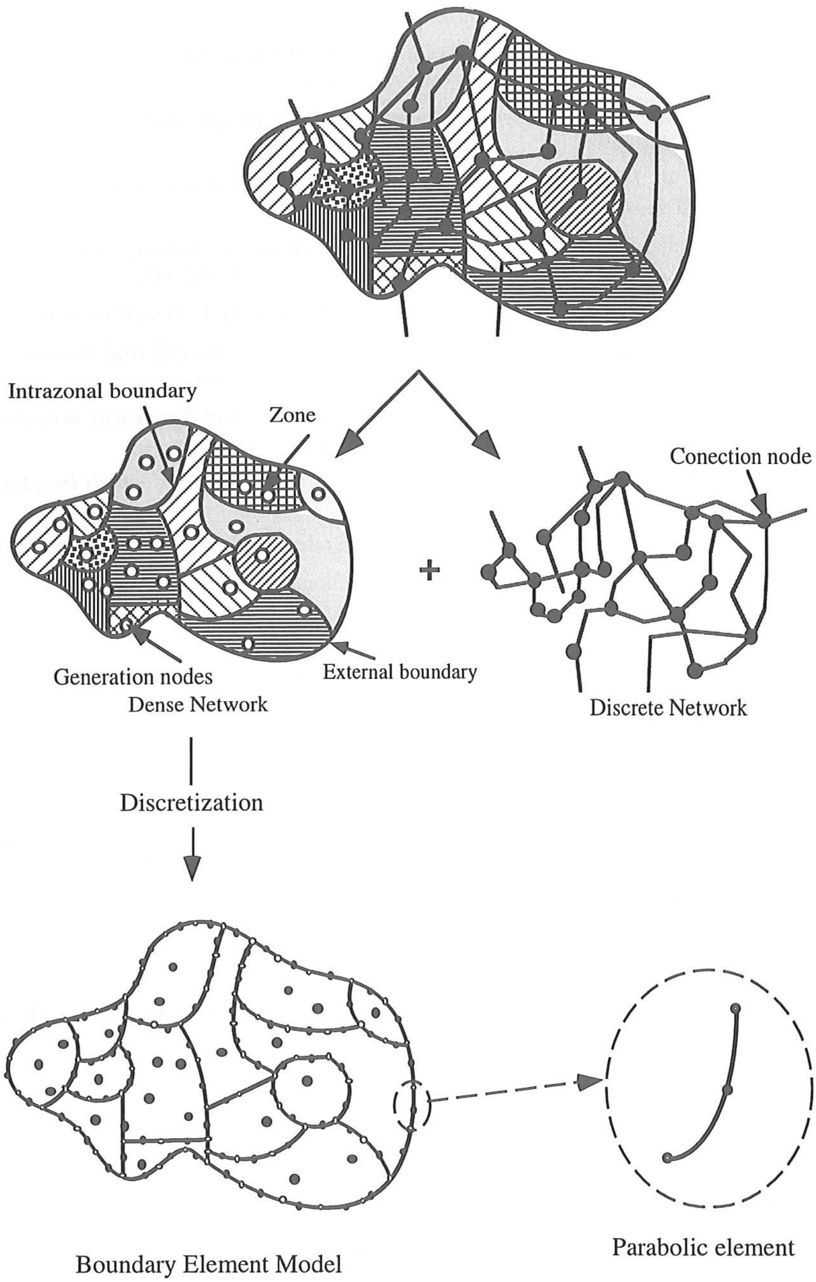

Figure 3 **Modelling process for urban regions**

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APPENDIX A

Approximation functions

An adequate expression for the approximation function of (19) is:

$$f_k(\mathbf{x}) = 9\mu r - 3r[(x - x_k)\bar{u}_x + (y - y_k)\bar{u}_y], \tag{A.1}$$

where r is the distance between pre-specified points (x_k, y_k) and boundary nodes (x,y). The above expression is obtained by substituting $\psi = r^3$ into (21). Besides, the derivative with respect to normal is:

$$\frac{\partial \psi}{\partial \mathbf{n}} = 3r[(x - x_k)n_x + (y - y_k)n_y]. \tag{A.2}$$

For functions r_k and s_k used in the approximations of $\nabla(\tilde{\mathbf{u}}\phi)$ and $\rho(\mathbf{x})$, respectively, the series:

$$1 + r + r^2 + \cdots + r^m, \tag{A.3}$$

could be used, although using $1 + r$ would be the simplex alternative.

Fundamental solution

The fundamental solution for the steady-state convection-diffusion equation (15) is given by:

$$\psi(\mathbf{x}, \xi) = \frac{1}{2\pi\mu} e^{-\frac{\bar{\mathbf{u}}\cdot\mathbf{r}}{2\mu}} [-\mu K_1(\mu r)\frac{\partial \mathbf{r}}{\partial \mathbf{n}} - \frac{\mathbf{u}_n}{2\mu} K_0(\mu r)], \tag{A.4}$$

where K_0 and K_1 are Bessel functions of second kind of order zero and one, respectively.