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TOPIC 18
ENVIRONMENT AND
SUSTAINABLE MOBILITY

OPTIMAL DEPARTURE SCHEDULING FOR THE MORNING RUSH HOUR WHEN CAPACITY IS UNCERTAIN

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Abstract

The system optimal morning rush hour departure rate is solved for one OD pair connected by a bottleneck with stochastic capacity. The departure rate increases over time and is independent of the right-hand tail of the capacity distribution. A time-varying toll supports the optimum as a user equilibrium.

INTRODUCTION

Dynamic traffic equilibrium models fall into four categories: deterministic user equilibrium, deterministic system optimum, stochastic user equilibrium and stochastic system optimum. Considerable research effort has gone into the development of algorithms that can solve for the associated traffic flows on networks of practical interest. Yet such algorithms remain elusive even for the deterministic models. Moreover, conceptual difficulties persist in how to model traffic behaviour over space and time, and in understanding how drivers behave under uncertainty.

Given this state of affairs, analytical models continue to have a role in developing intuition, and in helping to explain behaviour of more complex models. The Vickrey (1969) bottleneck model of the morning rush hour has been particularly useful in examining various aspects of travel behaviour, including departure time, route choice, elastic demand, driver heterogeneity and tolling. However, attention has generally been confined to nonstochastic environments.

Recently, the stochastic user equilibrium for the Vickrey model with a single route was solved analytically by Arnott et al. (1994). The purpose of this paper is to solve the stochastic system optimum for the same model. Together, the papers provide a conceptual framework for studying the effects of Electronic Road Pricing and Advanced Traveler Information Systems.

The paper is organized as follows. First the deterministic user equilibrium (DUE) and deterministic system optimum (DSO) of the Vickrey bottleneck model are reviewed. The following section develops and solves the stochastic system optimum (SSO), and indicates how it can be decentralized using a time-varying toll. A comparison is made with both the DSO, and the stochastic user equilibrium (SUE) derived in Arnott et al. (1994). The next section examines in detail a simple example that displays intuitive properties that are not transparent in the general specification. The conclusion provides a summary of results.

THE BOTTLENECK MODEL IN A DETERMINISTIC SETTING

The deterministic model is a variant of Vickrey's (1969) bottleneck queueing model of morning rush-hour traffic. It is described and analyzed in detail in Arnott et al. (1990). Only the essentials are given here.

In the model, N identical individuals drive, one per car, from home to work along a single route. Demand is independent of price, so that N is fixed. Road capacity is determined by a bottleneck with a maximum service rate s. If the arrival rate of drivers at the bottleneck exceeds s, a queue develops. Let r(t) denote the arrival rate of users at the bottleneck at time t, Q(t) the number of vehicles in the queue, and $\hat{t}$ the time at which the queue was last zero. Then $Q(t) = \int_{\hat{t}}^{t} r(\tau)d\tau - s(t-\hat{t})$. Time spent queueing is q(t) = Q(t)/s.

Without loss of generality it is assumed that travel time under free-flow conditions is zero, so that individuals reach the bottleneck as soon as they leave home, and once through the bottleneck arrive immediately at work. Hence a driver leaving at time t arrives at work at time $t_a(t) = t + q(t)$.

Individuals are assumed to have a common preferred arrival time at work, t*, and incur a schedule delay cost $D(t_a - t^*)$ if they arrive at t_a instead. t* can be thought of as the official starting time for work. It is assumed that D(·) is (weakly) convex, with D(0) = 0, D(x) > 0 for x≠0. In parts of the paper it will be assumed that D(·) has the piecewise linear form:

$$D(t_a - t^*) = \beta \operatorname{Max}[0, t^* - t_a] + \gamma \operatorname{Max}[0, t_a - t^*]. \quad (1)$$

The parameter β reflects the costs of arriving early, including the need to wake early, difficulties in scheduling household and other activities, and wasted time before work begins. Parameter γ includes any employer sanctions, or inconvenience to coworkers, from arriving late.

In addition to schedule delay costs, an individual departing at t incurs a queueing time cost $\alpha q(t)$. This cost includes gasoline consumption, vehicle depreciation, and the opportunity cost of time. Trip costs for a driver departing at t are the sum of queueing costs, schedule delay costs and any toll, $\tau(t)$:

$$C(t) = \alpha\, q(t) \;+\; D(t + q(t) - t^{*}) \;\; + \tau(t). \tag{2}$$

To assure that the departure rate is finite in the no-toll equilibrium it is necessary to assume that $\alpha + D'(\cdot) > 0$. This condition holds necessarily for late arrivals. For early arrivals, it requires that a minute of time spent queueing be more costly than an extra minute of time spent waiting for work to begin.

In the stochastic setting to be considered in the following section it will be convenient to work with normalized variables. To smooth the transition we introduce them here. Define $\rho(t) \equiv r(t)/N$ to be the normalized departure rate. ($\rho(t)dt$ is the fraction of drivers who depart during the time interval (t, t+dt).) Let R(t) be cumulative normalized departures. And let $\phi \equiv N/s$ be the time required for all drivers to pass through the bottleneck.

Deterministic User Equilibrium (DUE)

In the DUE, trip costs equal schedule delay plus travel time costs. This sum must be constant during the departure period. Figure 1 depicts such an equilibrium. Variables corresponding to the DUE are denoted by a superscript n. Variables corresponding to the DSO—considered below—are denoted by a superscript o. The first driver departs at time t_o^n. Encountering no queue, he reaches work immediately at a cost $C^n(t_0^n) = D(t_0^n - t^*)$ that includes only schedule delay. A driver departing at t > t_o^n incurs a cost $C^n(t) = \alpha q^n(t) = D(t + q^n(t) - t^*)$. In equilibrium, $C^n(t) = C^n(t_0)$ for all t in the departure period. Queueing time reaches a maximum at t_*^n for the driver who arrives on time.

The last driver departs at time t_e^n, and avoids queueing. So the first and last drivers both incur only schedule delay costs. This common cost must equal equilibrium trip cost, C^n:

$$C^n = D(t_0^n - t^*) \;=\; D(t_e^n - t^*). \tag{3}$$

Since the bottleneck operates at capacity throughout the travel period,

$$t_e^n - t_0^n = \phi\,. \tag{4}$$

Equations (3) and (4) can be solved for t_0^n, t_e^n and C^n. The solution values depend on the shape of the schedule delay cost function, but not on the unit cost of travel time, α. If schedule delay costs are given by (1), the solution is:

$$t_0^n = t^* - \frac{\gamma}{\beta+\gamma}\phi,\; t_e^n = t^* + \frac{\beta}{\beta+\gamma}\phi\,, \tag{5}$$

$$C^n = \delta\phi\,, \tag{6}$$

where $\delta \equiv \beta\gamma/(\beta+\gamma)$. Average schedule delay costs and average travel time costs are both $\delta\phi/2$.

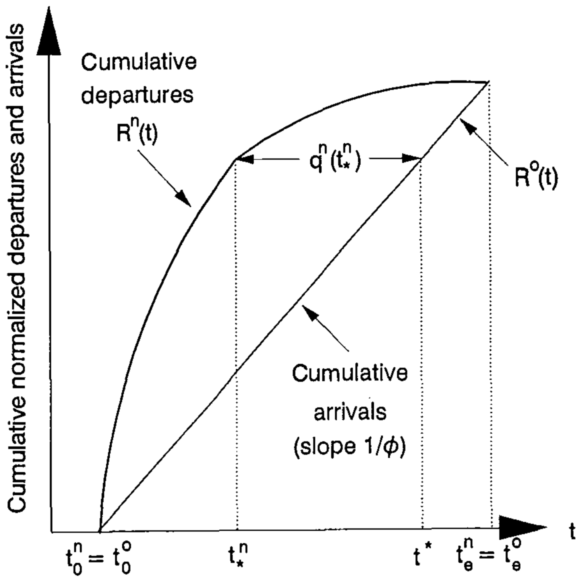

Figure 1 **No-toll deterministic user equilibrium and system optimum**

Deterministic System Optimum (DSO)

Queueing time is deadweight loss in the bottleneck model. The DSO is thus achieved by holding the departure rate at bottleneck capacity: $r^o(t) = s \Rightarrow \rho^o(t) = 1/\phi$ and $R^o(t) = (t - t_0^o)/\phi$. To minimize total schedule delay costs, t_0^o is chosen so that the first and last drivers incur the same schedule delay costs:

$$D(t_0^o - t^*) = D(t_e^o - t^*). \tag{7}$$

Furthermore, the bottleneck is at capacity throughout the rush hour:

$$t_e^o - t_0^o = \phi. \tag{8}$$

Because equations (7) and (8) are congruent with (3) and (4), the timing of the rush hour is the same as in the DUE. If schedule delay costs are given by (1), t_0^o and t_e^o are given by (5). Since total queueing costs are zero, average trip costs equal average schedule delay costs, which are the same as in the DUE. Hence

$$C^o = \frac{\delta}{2}\phi. \tag{9}$$

The DSO can be decentralized by using a time-varying toll, $\tau^o(t)$, that starts at zero at t_0^o and evolves at a rate equal in magnitude and opposite in sign to the change in schedule delay costs. Since this is precisely how queueing cost behaves in the DUE, $\tau^o(t) = \alpha q^n(t)$, $t \in [t_0^o, t_e^o]$.

THE BOTTLENECK MODEL UNDER UNCERTAINTY

Introduction

In reality, travel times are rarely constant, as is assumed in the deterministic model, but rather vary from day to day. Probably the most important source of variation are fluctuations in road capacity due to road repairs, construction, accidents, vehicle disablings, bad weather, snow removal, etc. Fluctuations are modeled here by treating $\phi = N/s$ as a random variable, with a cumulative distribution function $J(\phi)$, and a range $[\phi^{Min}, \phi^{Max}]$. Within this range, the probability density can be zero, and can also have mass points.

ϕ is assumed to fluctuate from day to day, but to remain constant during the travel period on a given day. This assumption is reasonable for capacity fluctuations due to road work, weather and possibly major truck accidents; less so for automobile accidents and disablings.

Stochasticity in ϕ creates a dilemma. If capacity is low, then a high departure rate will lead to a long queue. This can be avoided by choosing a conservative departure rate. But if capacity is high, the bottleneck will be underutilized, and aggregate schedule delay costs will be unnecessarily high. The optimal departure rate entails a balance between these two costs.

To find this balance, it is necessary to distinguish states in which a queue exists from states in which there is no queue. Let $\phi(t)$ denote the maximum ϕ at which no queue exists at time t, $t > t_0$. Suppose that cumulative departures, R(t), follow the time path shown in Figure 2. If ϕ^{Min} is realized, the normalized capacity of the bottleneck is $1/\phi^{Min}$. This is indicated by the slope of the steeper of the two rays emanating from the point $(t_0, 0)$. As drawn, R(t) is everywhere flatter than this ray; ie $\rho(t) < 1/\phi^{Min}$. So a queue never develops when $\phi = \phi^{Min}$. If, at the other extreme, ϕ^{Max} is realized, the normalized capacity of the bottleneck is $1/\phi^{Max}$, equal to the slope of the flatter ray from $(t_0, 0)$. R(t) lies everywhere above this ray, so that a queue exists when $\phi = \phi^{Max}$ for all $t > t_0$.

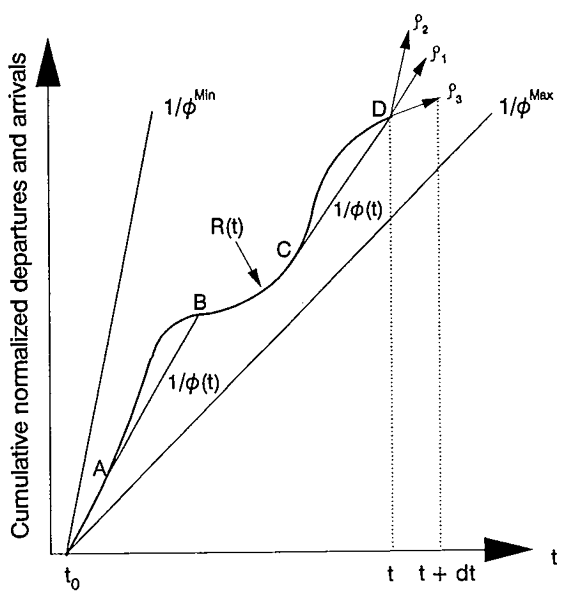

Figure 2 Queueing patterns in different states

Suppose, finally, that $\phi = \phi(t)$. A queue starts to build at point A, where the departure rate $\rho(t)$ reaches $1/\phi(t)$. The queue disappears at point B, starts again at C, and dissipates again at D and time t. For $\phi>\phi(t)$, a queue exists at t, whereas for $\phi<\phi(t)$, queueing ends before t, if indeed a queue develops at all. Hence, consistent with its definition, $\phi(t)$ is the largest ϕ for which there is no queue at time t.

What happens after time t depends on the departure rate. Three possible values, $\rho_1 = 1/\phi(t)$, $\rho_2 > 1/\phi(t)$, and $\rho_3 < 1/\phi(t)$, are shown in Figure 2. If for $t' \in (t,t+dt)$, $\rho(t') = \rho_1$, then $\phi(t+dt) = \phi(t)$; the critical ϕ does not change. If $\rho(t') = \rho_2$, then $\phi(t+dt) = 1/\rho_2$; the critical ϕ jumps to a lower value. Finally, if $\rho(t') = \rho_3$, the critical ϕ will decrease by an amount proportional to dt. In summary,

$$\lim_{dt \downarrow 0} \phi(t+dt) = \begin{cases} \phi(t) & \text{if } \rho(t') \leq 1/\phi(t) \\ \rho(t') & \text{if } \rho(t') \geq 1/\phi(t) \end{cases} \quad \text{for } t' \in (t, t+dt).$$

This in turn implies the following differential relationship between $\phi(t)$ and $\rho(t)$:

$$\frac{d\phi(t)}{d\rho(t)} = \begin{cases} -1/[\rho(t)]^2 & \text{if } \rho(t) = 1/\phi(t) \text{ and } \dot{\rho}(t) \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where the dot over a variable indicates a time derivative. The complication that $\phi(t)$ is not a differentiable function of $\rho(t)$ must be addressed when solving for the optimal $\rho(t)$, a problem addressed next.

The Stochastic System Optimum (SSO)

In this section, the problem of choosing a time path for $\rho(t)$ that supports the SSO is formulated using the theory of optimal control. The objective function is taken to be aggregate expected trip costs. This choice follows from two prior assumptions, and one new one: (1) The number of trips, and hence the gross benefits of travel, are given; (2) individuals are identical, so that there is no reason to discriminate between them; (3) individuals are risk neutral with respect to trip costs.

To simplify notation, superscript o denoting the SSO will be omitted except where desired for emphasis or later reference. Let $q(t,\phi) \geq 0$ denote queueing time at time t given ϕ. The control problem can then be stated as follows:

$$\min_{t_0, t_e, \rho(t)} \int_{t_0}^{t_e} \left\{ D(t-t^*)\, J(\phi(t)) + \int_{\phi(t)}^{\phi^{Max}} \left[D(t+q(t,\phi)-t^*) + \alpha q(t,\phi) \right] dJ(\phi) \right\} \rho(t)\, dt \tag{11}$$

subject to (10) and the constraints:

$$\dot{q}(t,\phi) = \begin{cases} \phi\rho(t) - 1 & \text{if } \phi \geq \phi(t) \\ 0 & \text{if } \phi \leq \phi(t) \end{cases} \qquad [\text{multiplier } \lambda(t,\phi) \geq 0] \tag{a}$$

$$\dot{R}(t) = \rho(t) \qquad [\text{multiplier } \mu < 0], \tag{b}$$

$$R(t_0) = 0, \quad R(t_e) = 1\,. \tag{c}$$

The objective in (11) is an integral over the departure period $[t_0, t_e]$ of expected trip costs of individual drivers, multiplied by the normalized rate at which they depart, $\rho(t)$. The first term in the integrand for t accounts for realizations of $\phi \leq \phi(t)$, for which there is no queue, and hence for which drivers incur only schedule delay costs. The probability of no queue is $J(\phi(t))$. The second term in the integrand accounts for states in which queueing does occur.

Constraint (a) specifies the equation of motion for queueing time in each state. A separate multiplier, or adjoint variable, $\lambda(t,\phi)$, is defined for each ϕ. $\lambda(t,\phi)$ can be interpreted as the shadow cost of queueing time at time t, given ϕ. Constraint (b) accounts for the evolution of the state variable R(t). Because an exogenous increase in R(t) would reduce the number of drivers left to travel, the shadow (ie marginal system) cost of R(t), given by multiplier μ, is negative. Initial and final conditions on R(t) are given in (c).

The last driver, who departs at t_e, arrives at work at time $t_e + q(t_e,\phi)$, which is after t_e if $\phi > \phi(t_e)$. But since everyone has now departed, the shadow cost of queueing time is zero, regardless of ϕ. Hence

$$\lambda(t_e,\phi) = 0, \quad \text{for all } \phi \in [\phi^{\text{Min}},\phi^{\text{Max}}]. \tag{12}$$

Derivation of the optimal departure schedule

The Hamiltonian for (11) is (13)

$$H = \left\{ D(t-t^*)\, J(\phi(t)) + \int_{\phi(t)}^{\phi^{Max}} [D(t+q(t,\phi)-t^*) + \alpha q(t,\phi)]\, dJ(\phi) + \mu \right\} \rho(t) + \int_{\phi(t)}^{\phi^{Max}} \lambda(t,\phi)[\phi\rho(t)-1] dJ(\phi).$$

Necessary conditions for a minimum are

$$\frac{\partial H}{\partial \rho(t)} = D(t-t^*)\, J(\phi(t)) + \int_{\phi(t)}^{\phi^{Max}} [D(t+q(t,\phi)-t^*) + \alpha q(t,\phi)]\, dJ(\phi) + \mu$$

$$+ \int_{\phi(t)}^{\phi^{Max}} \lambda(t,\phi)\phi dJ(\phi) - \lambda(t,\phi(t))[\phi(t)\rho(t)-1]\frac{dJ(\phi(t))}{d\phi}\frac{d\phi(t)}{d\rho(t)} = 0\,, \tag{14}$$

$$\dot{\lambda}(t,\phi) = -\frac{\partial H}{\partial q(t,\phi)} = \begin{cases} -\rho(t)[D'(t+q(t,\phi)-t^*)+\alpha] & \text{if } \phi \geq \phi(t) \\ 0 & \text{otherwise} \end{cases}, \tag{15}$$

$$H(t_0) = 0, \quad H(t_e) = 0. \tag{16}$$

Condition (14) governs the optimal departure rate. Condition (15) governs the evolution of the adjoint variables. By assumption, $D'(\cdot)+\alpha>0$, so that $\dot{\lambda}(t,\phi) \leq 0$: the shadow cost of queueing in any state decreases monotonically over time. This is because the number of drivers still at home, and who will be delayed by the queue, is declining. Finally, (16) gives transversality conditions for t_0 and t_e, which are unconstrained.

Given condition (10), the last term in (14) is zero whether or not $\rho(t) = 1/\phi(t)$. Hence (14) can be written:

$$D(t-t^*)\, J(\phi(t)) + \int_{\phi(t)}^{\phi^{Max}} [D(t+q(t,\phi)-t^*) + \alpha q(t,\phi) + \lambda(t,\phi)\phi] dJ(\phi) + \mu = 0\,. \tag{17}$$

As our first result we have (proofs of all propositions are given in Lindsey 1994):

Proposition 1: The optimal departure rate, $\rho^o(t)$, is weakly increasing over the departure period; hence the cumulative departures schedule, $R^o(t)$, is convex.

The intuition behind Prop. 1 is that the cost of queueing in any state, as measured by $\lambda(t,\phi)$, diminishes over time as the number of drivers yet to depart decreases. Hence, the initial departure rate is chosen conservatively, then gradually liberalized as time passes.

The timing of departures is described in:

Proposition 2: In the SSO, departures occur continuously over the time interval $[t_0^o, t_e^o]$, where $t_0^o < t^*$, $t_e^o \geq t^*$.

Proposition 3: The marginal system cost of a driver is $-\mu = D(t_0^o - t^*)$.

To see this, note that since a driver can leave just before t_0^o without imposing any costs on others, the system cost of an extra driver is the private travel cost of a driver at t_0^o , $D(t_0-t^*)$. Adding a driver increases required cumulative departures by one, at a cost $-\mu$. Hence $-\mu = D(t_0^o - t^*)$.

The initial departure rate is specified in

Proposition 4: $\rho(t_0^o) = 1/\phi^{Max}$.

If the initial departure rate exceeded $1/\phi^{Max}$, queueing would begin immediately for large values of ϕ. The system cost of the first drivers would exceed their private costs by an appreciable amount, and it would be cheaper to have these drivers leave just before t_0^o , a contradiction.

The terminal departure rate is given by

Proposition 5:

$$\rho(t_e^o) = \left(J^{-1}\left[\frac{\alpha}{\alpha + D'(t_e^o - t^*)} \right] \right)^{-1} . \tag{18}$$

Equation (18) involves t_e^o, which cannot generally be expressed in closed form. But if schedule delay costs are given by (1), equation (18) reduces to:

$$\rho(t_e^o) = \left(J^{-1}\left[\frac{\alpha}{\alpha + \gamma} \right] \right)^{-1} . \tag{19}$$

As $\rho(t)$ rises toward a maximum at t_e^o, $\phi(t)$ falls to a minimum at the $\alpha/(\alpha + D'(t_e^o - t^*))$ fractile of the distribution of ϕ, as can be seen from (18). The distribution of ϕ below this fractile has no influence on either the departure rate or on travel costs. Hence capacity investments or traffic management policies that improve capacity only in the most favourable states are worthless.

A departure schedule with properties consistent with Props. 1-5 is shown in Figure 3. The departure rate is held to $1/\phi^{Max}$ for an initial period. At t_q it starts to rise, and queueing develops in the least favourable (high ϕ) states. After t_{σ}^o , drivers arrive late in some states. Departures cease at $t_e^o \geq t^*$ with the departure rate at $1/\phi(t_e^o)$, where $\phi(t_e^o) \in [\phi^{Min}, \phi^{Max}]$.

To summarize: in the SSO queueing occurs in unfavourable states, whereas capacity is underutilized in favourable states. This contrasts with the DSO, in which the departure rate is held constant at (known) capacity, and neither queueing nor underutilization of capacity occurs.

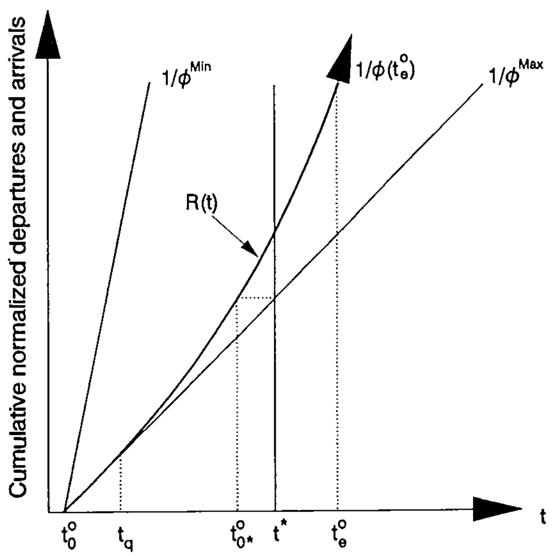

Figure 3 **A departure schedule for the SSO**

Stochastic User Equilibrium (SUE)

The SUE was studied by Arnott et al. (1994) for the case of linear schedule delay costs. Here we give an abridged treatment, and focus on comparison with the SSO. As illustrated in Figure 1, given a convex schedule delay cost function and certainty, the departure rate in the DUE is monotonically *decreasing* with time. This turns out also to be true of the SUE when no tolls are imposed:

Proposition 6: In the no-toll SUE the departure rate $\rho^n(t)$ is weakly decreasing over the departure period; hence cumulative departures $R^n(t)$ are concave.

Prop. 6 stands in contrast to Prop. 1, which states that the departure rate is monotonically *increasing*. To see why $\rho^n(t)$ decreases over time note that as time passes, the expected rate of reduction in schedule delay costs from postponing departure decreases, and eventually becomes negative. For expected private trip costs to remain constant, expected queueing costs must increase at a decreasing rate, and eventually decline.

The timing of departures, which is qualitatively the same as in Prop. 2 for the SSO, is described in:

Proposition 7: In the no-toll SUE, departures occur continuously over the time interval $[t_0^n, t_e^n]$, where $t_0^n < t^*$, $t_e^n \geq t^*$.

Given the qualitative similarity between the timing of departures in the no-toll SUE and the SSO, one may ask whether the departure periods coincide. Towards answering this, note that in the no-toll SUE, the expected private costs of the first and last drivers must be equal. In the SSO, their expected *system* costs must be equal. Now the system and private costs of the first driver are the same in the SSO because queueing is initially prevented. Moreover, since the shadow cost of queueing vanishes at t_e^o by (12), the system and private costs of the last driver are also the same. Hence in both the no-toll SUE and the SSO, the private costs of the first and last drivers are equal.

This might suggest that the departure periods are also equal. But this is not the case, for the departure rate schedules differ, and hence too the distribution of queueing times faced by the last drivers in the two regimes. The fact that departure times can differ will be confirmed by example in the next section.

To further characterize the no-toll SUE, and to facilitate comparison with the SSO, we now assume that schedule delay costs are piecewise linear, as given in (1). Following Arnott et al. (1994) define

$$\tilde{\phi} \equiv J^{-1}\left[\frac{\alpha}{\alpha+\gamma}\right], \quad \hat{\phi} \equiv \frac{1}{1-J(\tilde{\phi})}\int_{\tilde{\phi}} \phi dJ(\phi) = \frac{\alpha+\gamma}{\gamma}\int_{\tilde{\phi}} \phi dJ(\phi). \tag{20}$$

$\tilde{\phi}$ is the $\alpha/(\alpha+\gamma)$ fractile of the $J(\cdot)$ distribution, and $\hat{\phi}$ is the mean of ϕ for $\phi \geq \tilde{\phi}$. The following result is proved in Arnott et al. (1994).

Proposition 8: Let the schedule delay cost function be (1). Then

a) If

$$\tilde{\phi} \geq \frac{\gamma}{\beta+\gamma}\hat{\phi} \tag{21}$$

then

$$t_0^n = t^* - \frac{\delta}{\beta}\hat{\phi}, \quad t_e^n = t_0^n + \tilde{\phi} > t^*. \tag{22}$$

Expected trip costs are

$$C^n = C^n(t_0^n) = \beta(t^* - t_0^n) = \delta\hat{\phi}. \tag{23}$$

b) If (21) is not satisfied, then $t_e^n = t^*$, and $t_0^n = t^* - \phi_0$, where $\phi_0 > \tilde{\phi}$ is the unique solution of the following implicit equation

$$J(\phi_0) + \frac{1}{\phi_0}\int_{\phi_0}^{\phi^{Max}} \phi dJ(\phi) - \frac{\alpha+\beta+\gamma}{\alpha+\gamma} = 0. \tag{24}$$

Expected trip costs are

$$C^n = C^n(t_0^n) = \beta\phi_0. \tag{25}$$

Prop. 8 reveals that whether departures continue after t* in the no-toll SUE depends on the extent of variation in ϕ. If the distribution is not too spread out ((21) holds) then departures continue; otherwise they stop. The distribution of ϕ thus affects the *qualitative* properties of the no-toll SUE. However, small values of ϕ in the distribution may have no effect. To see this, suppose first that (21) is satisfied. Then, according to (23), expected trip costs are proportional to $\hat{\phi}$: the mean value of ϕ above the $\alpha/(\alpha+\gamma)$ fractile of the $J(\cdot)$ distribution. The distribution of ϕ below this fractile has no effect on trip costs. According to (19), the SSO has the same property.

If (21) is not satisfied, costs are given by (25), which is independent of the distribution of ϕ below ϕ_0. Since $\phi_0 > \tilde{\phi}$, costs are invariant to a larger proportion of the distribution than when condition (21) holds.

It is evident that the no-toll SUE and the SSO may or may not be functions of the same range of the ϕ distribution. But in either case, the no-toll SUE is invariant to the most favourable realizations of ϕ. Once again, a policy that enhanced capacity only in the most favourable states would be ineffective.

Decentralization of the SSO with a time-varying toll

In this section, we return to the general schedule delay cost specification and show that the SSO can be supported as a SUE by use of a time-varying toll. Drivers are assumed to pay the toll when they enter the commuting corridor.

In the deterministic setting considered in earlier, the optimal time-varying toll equals queueing time in the no-toll DUE: $\tau^o(t) = \alpha q^n(t)$. The toll internalizes the costs drivers impose on other drivers, and assures that the equilibrium departure rate equals capacity.

Following the same principle, the time-varying toll with uncertainty is set equal to the difference between expected system and private travel costs. In light of (17), the requisite toll is

$$\tau^o(t) = \int_{\phi(t)}^{\phi^{Max}} \lambda(t,\phi)\phi dJ(\phi); \tag{26}$$

that is, the toll equals the expected increase in travel costs imposed on later drivers. Now, given Prop. 4, $\phi(t_0^o) = \phi^{Max}$, which implies that $\tau^o(t_0^o) = 0$. And given (12), $\tau^o(t_e^o) = 0$. Hence the toll is zero at the beginning and the end of the departure period, just as in the deterministic setting. During the departure interval, the toll is positive, and is a concave function of time. These results are summarized in:

Proposition 9: The time-varying toll that decentralizes the SSO is a concave function, $\tau^o(t)$, such that $\tau^o(t_0^o) = 0$, $\tau^o(t_e^o) = 0$, $\tau^o(t) > 0$ for $t \in (t_0^o, t_e^o)$.

When the optimal time-varying toll is applied, the SSO is supported as a SUE, and expected private trip costs are constant during the departure period. What about *dispersion* in trip costs? It is apparent from Figure 3 that after t_q both expected queueing delay and the range of possible queueing delays increase with t. This suggests that trip costs also become increasingly dispersed. It is straightforward, but tedious, to show that this is indeed true in the sense of second-order stochastic dominance. Stated formally, we have:

Proposition 10: Let $\tilde{C}(t)$ be the random variable denoting trip costs at time t in the SSO. Suppose $t_0^o \le t < t' \le t_e^o$. Then $\tilde{C}(t)$ has second-order stochastic dominance over $\tilde{C}(t')$.

Prop. 10 suggests that drivers who are risk-averse with respect to trip costs will prefer to depart early in the SSO. However, this is not necessarily true of the no-toll SUE, as demonstrated by example in the next section.

AN EXAMPLE

In this section we explore the properties of a simple example. The example serves three purposes. First, it illustrates properties of the SSO and no-toll SUE that were derived in earlier. Second, it exhibits additional properties of the two regimes that cannot be deduced from the general model, but are intuitively reasonable. Third, it shows that expected cost reductions from implementing the SSO with a time-varying toll can be comparable to the gains from implementing the corresponding DSO when ϕ is known.

Specification

For the example, schedule delay costs are given by (1) and ϕ has a two-point distribution: $\phi = \phi_1$ with probability $1-\pi$, $\phi = \phi_2 > \phi_1$ with probability π. Hence the cumulative distribution of ϕ is

$$J(\phi) = \begin{cases} 0 & \phi < \phi_1 \\ 1-\pi & \phi_1 \le \phi < \phi_2 . \\ 1 & \phi \ge \phi_2 \end{cases} \tag{27}$$

State 1, in which $\phi = \phi_1$, can be thought of as "normal" travel conditions, in which road capacity is at its design level. State 2 entails reduced capacity due to accidents, vehicle disablings, bad weather, etc.

Stochastic System Optimum (SSO)

The optimal departure schedule for the SSO is described in:

Proposition 11: Let D(·) be given by (1), and J(ϕ) by (27). Then

(a) If

$$\pi \le \gamma / (\alpha + \gamma), \tag{28}$$

then

$$\rho^o(t) = \begin{cases} 1/\phi_2, & \text{for } t \in (t_0^o, t_{21}^o) \\ 1/\phi_1, & \text{for } t \in (t_{21}^o, t_e^o) \end{cases}, \tag{29}$$

where

$$t_0^o = t^* - \frac{\gamma}{\beta+\gamma}\left[1 - \frac{\beta}{\gamma}\frac{[(1-\pi)(\alpha+\gamma)-\alpha](1-\pi)(1-\sigma)}{(1-\pi)(\beta+\gamma)-[(1-\pi)(\alpha+\gamma)-\alpha](1-\pi+\pi\sigma)}\right]\phi_2, \tag{30}$$

$$t_{21}^o = t^* - \frac{\beta}{\beta+\gamma}\left[\frac{(1-\pi)(\alpha+\gamma)-\alpha}{(1-\pi)(\beta+\gamma)-[(1-\pi)(\alpha+\gamma)-\alpha](1-\pi+\pi\sigma)}\right]\phi_1, \tag{31}$$

$$t_e^o = t^* + \frac{\beta}{\beta+\gamma}\left[\frac{\alpha-(1-\pi)(\alpha-\beta)}{(1-\pi)(\beta+\gamma)-[(1-\pi)(\alpha+\gamma)-\alpha](1-\pi+\pi\sigma)}\right]\phi_1. \tag{32}$$

Expected trip costs are

$$C^o = \frac{\delta}{2}(t^* - t_0^o). \tag{33}$$

(b) If condition (28) is not satisfied, then

$$\rho^o(t) = 1/\phi_2, \ \text{for } t \in (t_0^o, t_e^o), \tag{34}$$

where

$$t_0^o = t^* - \frac{\gamma}{\beta+\gamma}\phi_2, \ \ t_e^o = t^* + \frac{\beta}{\beta+\gamma}\phi_2; \tag{35}$$

$$C^o = \frac{\delta}{2}\phi_2. \tag{36}$$

Part (a) of Prop. 11 applies if capacity reduction is not too likely. The departure rate function behaves qualitatively as described for the general model earlier. The departure rate begins at $\rho = 1/\phi^{Max} = 1/\phi_2$. At $t^o_{21} \leq t^*$, it is raised to $1/\phi_1$, where it remains thereafter. The higher the probability of normal capacity the sooner the departure rate is raised. Queueing occurs only in state 2, and after t^o_{21}.

If condition (28) fails, part (b) of Prop. 11 applies. The departure rate is restricted to $1/\phi_2$ throughout the travel period. Both the timing of departures and travel costs are the same as if state 2 were *certain*. This illustrates the possibility, noted in Section 3, that the departure rate is independent of favourable states.

In the event that $\pi=\gamma/(\alpha+\gamma)$, the departure rate after t* can take any value between $1/\phi_1$ and $1/\phi_2$. To see this, suppose that the departure rate is increased from ρ to ρ', with $1/\phi_1 \leq \rho<\rho'\leq 1/\phi_2$. If $\phi=\phi_1$, the perturbation will result in a saving of late arrival costs equal to γ times the reduction in late time. If $\phi=\phi_2$, the perturbation will raise travel time costs by α times extra queueing time. Since the reduction in late time if $\phi=\phi_1$, equals the amount of extra queueing time if $\phi=\phi_2$, the expected benefit is positive if and only if $(1-\pi)\gamma>\alpha\pi$; ie if $\pi<\gamma/(\alpha+\gamma)$. If $\pi=\gamma/(\alpha+\gamma)$, any perturbation of the departure rate after t* within the interval $[1/\phi_1, 1/\phi_2]$ will leave total expected travel costs unchanged.

The effect of uncertainty on the timing of departures is described in:

Proposition 12: Let D(·) be given by (1), and J(ϕ) by (27). Let $(t^o_0(0), t^o_e(0))$ denote the departure interval when $\pi=0$. Then $t^o_0 < t^o_0(0)$. If $\pi>\gamma/(\alpha+\gamma)$, then $t^o_e > t^o_e(0)$. If $\pi<\gamma/(\alpha+\gamma)$ then $t^o_e < t^o_e(0)$.

Prop. 12 shows that in the face of possible capacity reductions, it is optimal to advance the start of the rush hour. However, the last departure may be earlier or later than under certainty.

No-toll Stochastic User Equilibrium (SUE)

The no-toll SUE can be derived using Prop. 8. Even for the simple two-point distribution of ϕ, six departure rate and queueing patterns can occur. These are illustrated in Figure 4, in conjunction with the corresponding SSO departure schedules. (A categorization of the various cases, as well as parameter values used to generate the examples, are given in Lindsey 1994.) Which of the six cases occurs depends on the values of β/α, γ/α, ϕ_1/ϕ_2 and π. The domains of each case in terms of ϕ_1/ϕ_2 and π are identified in Figure 5, using values of $\beta/\alpha= 0.61$ and $\gamma/\alpha = 2.38$ estimated by Small (1982, Table 2, model.1).

The six cases are defined by three criteria:

a) Whether $t^n_e > t^*$ or whether $t^n_e = t^*$. From Prop. 8, $t^n_e > t^*$ when condition (21) is satisfied. This is true of cases 1, 2, 3 and 4.

b) Whether drivers continue to depart after t* if no queue exists in state 1. This is true of cases 1, 2 and 3.

c) How long queueing persists in state 1. In cases 1 and 4, queueing lasts until after t*. In cases 2 and 5, it stops before t* . And in cases 3 and 6, a queue never develops in state 1.

Criterion b) is satisfied if and only if $\pi>\gamma/(\alpha+\gamma)$. By Prop. 11, the same condition determines whether the optimal departure rate remains at $1/\phi_2$ throughout the departure period. Hence in cases 1, 2 and 3, $\rho^o(t)$ remains at $1/\phi_2$, whereas in cases 4, 5 and 6, it increases to $1/\phi_1$ at $t^o_{21} < t^*$.

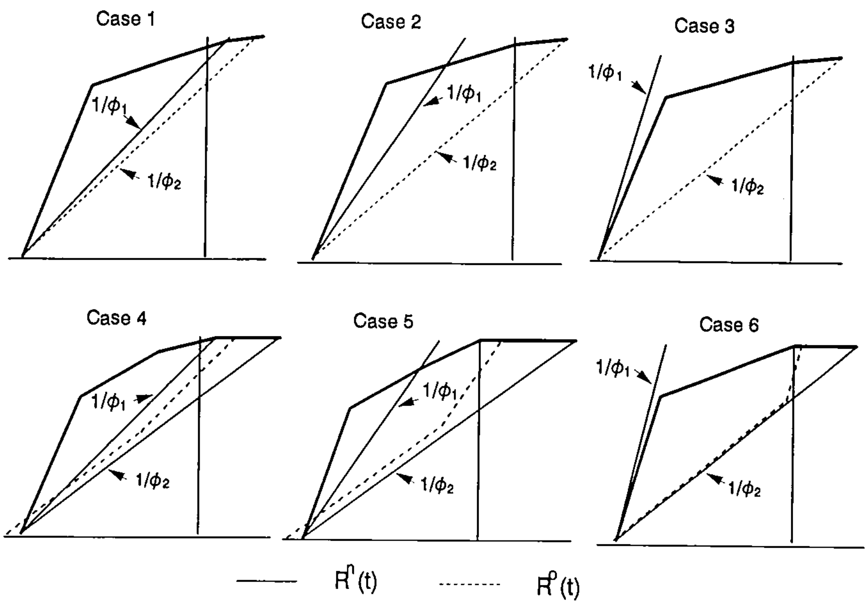

Figure 4 Departures in the no-toll SUE and SSO (two point distribution of Ø)

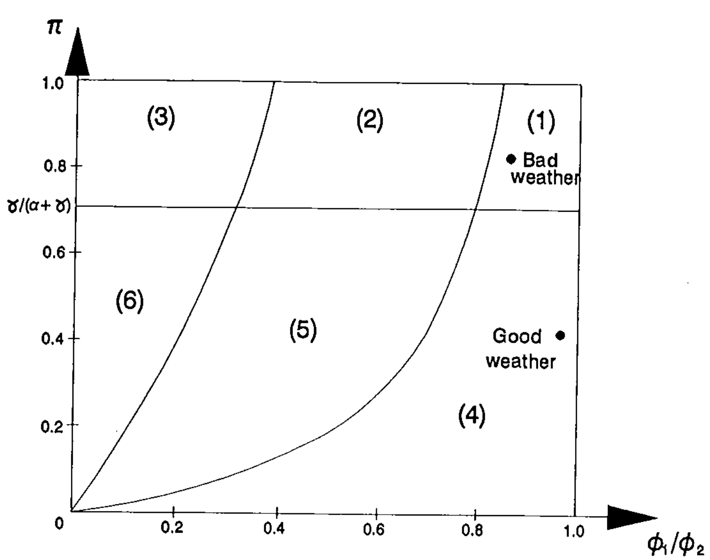

Figure 5 Parameter regions for the six no-toll SUE cases

Departures in the no-toll DUE and DSO occur over the same time interval, given by equation (5). For the no-toll SUE and SSO, the departure periods coincide for cases 1, 2 and 3. To see why, note that in these cases the last driver escapes queueing in both states. Naturally, this is also true for the first driver. Furthermore, the system costs of travel for the first and last drivers coincide with their private costs. Hence the schedule delay costs of the two drivers must be equal in both the no-toll SUE and SSO. This dictates that initial and final departure times coincide in the two regimes.

Whereas departures occur over the same time period in cases 1, 2 and 3, in cases 4, 5 and 6 they begin earlier and end later in the SSO. Because departures are more spread out in the SSO, the average utilization rate of capacity is lower. However, the *maximum* departure rate for the SSO can exceed the maximum departure rate in the no-toll SUE. This is evident in Figure 4 for Case 6.

The six cases differ in how variability in queueing time and trip costs evolve over time. In Cases 1, 2 and 3, the last driver escapes queueing and thus experiences no variation in travel time or costs. The same is of course true of the first driver. Hence dispersion in travel time and trip costs is zero at the beginning and end of the rush hour. This contrasts with the SSO, for which (per Prop. 10) dispersion increases throughout the departure period.

The behaviour of Cases 4, 5 and 6 is not obvious from Figure 4. It is straightforward to show algebraically that variability in travel time increases monotonically in Case 4, but can decrease in Cases 5 and 6. It is also possible to show that for all three cases variability in trip costs increases in the sense of second-order stochastic dominance. As far as empirical evidence, Bates et al. (1987) state: "normal ranges of variability are wider when reported by travellers who usually depart at later times of the morning". This is implied by Case 4, and consistent with Cases 5 and 6, but inconsistent with Cases 1, 2 and 3.

Consider finally the effect of imposing the optimal time-varying toll on drivers' expected trip costs. In the deterministic setting, tolling leaves private trip costs unchanged because it does not affect the departure period. This remains true for Cases 1, 2 and 3. But in Cases 4, 5 and 6, for which $t_0^o < t_0^n$, drivers are *worse off.* This follows from the fact that in the no-toll SUE the first driver incurs a cost $C^n = \beta(t^* - t_0^n)$, whereas in the optimal toll equilibrium the first driver incurs a cost $\beta(t^* - t_0^o) + \tau(t_0^o) = \beta(t^* - t_0^o)$. To overcome driver opposition, partial rebate of toll revenue or some other compensation might be necessary.

An "empirical" example

While the model is highly stylized, it is nevertheless worth asking which of the six cases are empirically plausible. The answer depends on a variety of factors, including those that affect β/α and γ/α, such as work hour flexibility, and those that affect ϕ_1/ϕ_2 and π, such as weather conditions, road design, number of driving lanes, speed limits and length of the commute (which affects the probability of accidents). Probability distributions of ϕ for "good weather" and for "bad weather" are given in Table 1. These are constructed using data from various empirical studies; see Lindsey (1994).

Table 1 Probability distribution of ϕ [hours]

Weather	ϕ_1	π	ϕ_2	ϕ_1/ϕ_2
Good	2	0.41	2.0967	0.9539
Bad	2.38	0.84	2.8222	0.8433

In good weather, the rush hour lasts two hours under normal conditions. Capacity reduction occurs with probability 0.41, with the effect of increasing time to traverse the bottleneck by about six minutes. In bad weather, travel time is increased by 19% under undisrupted conditions due to poor traction and/or limited visibility. Probability of capacity reduction rises to 0.84. Capacity conditional on an incident is 84.3% of its level without incidents, and the rush hour lasts 49 minutes longer than with good weather and no incidents.

The good weather capacity distribution occurs at the point labelled "Good weather" in Figure 5, and belongs to Case 4. Departures end at the same time as queueing in state 1, some time after t*. The bad weather distribution, labelled "Bad weather", falls into Case 1. This differs from Case 4 in that departures continue on until queueing ends in state 2.

Expected trip costs in the no-toll SUE and SSO of the example are given in panel (a) of Table 2. Costs are higher in bad weather than good weather because of the compound effect of poorer driving conditions and higher incident rates. Costs in the no-toll SUE are higher by $2.467 per trip in good weather, and by $3.33 in bad weather. These are the potential gains from implementation of electronic road pricing.

Table 2 Trip costs for the numerical examples [$]

(a) ϕ unknown				(b) ϕ known			
Weather	c^n	c^o	$c^n - c^o$	Weather	c^n	c^o	$c^n - c^o$
Good	4.99	2.52	2.467	Good	4.95	2.47	2.473
Bad	6.66	3.33	3.33	Bad	6.52	3.26	3.26

For purposes of comparison, travel costs in the two regimes are also calculated on the assumption that ϕ is known with certainty (both by drivers and the planner) each day. As shown in panel (b) of Table 2, both travel costs and the potential benefits from tolls are very similar to their values under uncertainty.

SUMMARY AND CONCLUDING REMARKS

In this paper we have derived the Stochastic System Optimum (SSO) departure rate for the Vickrey bottleneck model of morning rush hour. The paper complements Arnott et al. (1994), who solved for the no-toll Stochastic User Equilibrium (SUE). The main conclusions are as follows:

1. The optimal departure rate is nondecreasing over the departure period (Prop. 1). This contrasts with the no-toll SUE, in which the departure rate is nonincreasing (Prop. 6).
2. The initial departure rate equals the minimum possible capacity, thereby preventing queueing in any state (Prop. 4).
3. The maximum departure rate, which occurs at the end of the departure period, is set at a fractile of the capacity distribution that depends on the relative unit costs of travel time and late arrival (Prop. 5). Both the SSO departure rate schedule and expected travel costs are thus invariant to capacity realizations in the right-hand tail of the distribution. The same is true of the no-toll SUE (Prop. 8).
4. The SSO can be decentralized by applying a toll at the entrance to the travel corridor. The toll is a concave function of time, and zero at the beginning and end of the departure period (Prop. 9). Trip costs inclusive of the toll become increasingly dispersed with departure time (Prop 10).
5. In the case of a two-point distribution of capacity, the SSO departure period begins either at the same time or earlier than in the no-toll SUE. Hence, if a time-varying toll is used to support the SSO, expected private trip costs for drivers are equal to or greater than without tolling.

One feature of the model deserves comment: the assumption that capacity is constant during the travel period on a given day. This is not very realistic for capacity fluctuations due to accidents, and several recent studies of advanced traveler information systems have allowed for temporary capacity reductions. However, these studies have either focused on different aspects of traffic behaviour, or been forced to adopt other stringent assumptions. For example, Al-Deek and Kanafani (1993) focus on route choice during off-peak travel times, rather than departure time choice during the peak. And they treat the inflow rate of vehicles into the travel corridor as constant and exogenous, and include only trip duration in travel costs.

El-Sanhouri (1994) adopts what might be called a "quasi-equilibrium" approach in which drivers choose departure times and routes to minimize expected trip costs, but ignoring the possibility of accidents. The tolls he considers are step tolls, optimized on the assumption that capacity reductions do not occur, rather than continuously time-varying tolls conditioned on the probability distribution of capacity, as here. Finally, he uses simulation rather than analytical methods.

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